Arithmetic proof of the addition theorem of velocities in the special theory of relativity

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§1. Philosophical foreword

Many mathematicians/physicists would think that the relation between mathematics and physics are extremely deep and that these two sciences were once a single branch of the tree of knowledge (See Atiyah [11]). In fact, large parts of mathematics, including geometry and analysis, were developed primarily in connection with physics. Furthermore, present-day physics is employing some of the most abstract pure mathematics. Now suppose that you must write a book whose title is "Mathematics and Physics". Even though you know both mathematics and physics, it would be difficult to explain what the "and" means in the title (See Manin [12]).

Let us recall the classic philosophy "pre-established harmony" due to G.W. Leibniz ("Monadology" in [13]).

33. There are also two kinds of truth : those of reasoning, and those of fact. Truths of reasoning are necessary, and their opposite is impossible ; those of fact are contingent, and their opposite is possible. When a truth is necessary, the reason for it can be found by analysis, by resolving it into simpler ideas and truths until we arrive at the basic ones.

34. Thus mathematicians use analysis to reduce speculative theorems and practical canons to definitions, axioms, and postulates.

35. And finally there are the simple ideas, which cannot be given a definition ; and there are axioms and postulates - in a word, basic principles, which can never be proved, but which also have no need of proof : there are identical propositions, the opposite of which contains an explicit contradiction.

51. But in simple substances this influence of one monad over another is only ideal, and it can have its effect only through the intervention of God : in the sense that God's ideas one monad requires of God, and with reason, that he take account of when he organizes the others at the very beginning of things. Because, as one created monad could never have a physical influence over the interior of another, this is the only way in which one monad can depend on another.

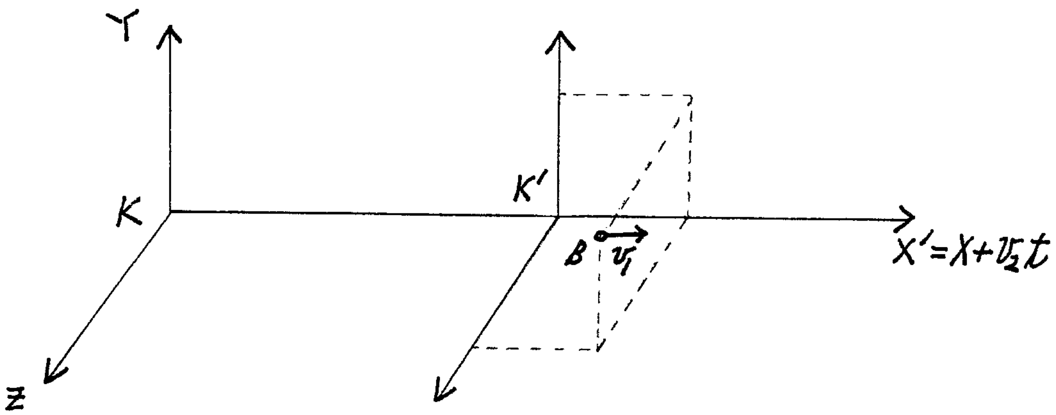
78. These principles gave me a way of providing a natural explanation of the union, or the conformity, of the soul with the organic body. The soul follows out its own laws, just as the body too follows its own. They are in agreement in virtue of the fact since they are all representations of the same universe, there is a pre-established harmony between all substances.

79. Soul acts according to the laws of final causes, through appetition, ends, and means. Bodies act according to the laws of efficient causes, or of motions. And these realms, that of efficient causes and that of final causes, are in mutual harmony.

87. Just as we earlier established a perfect harmony between two natural realms, the one of efficient causes and the other of final causes, so we must also point out here another harmony, between the physical realm of nature, and the moral realm of grace ; that is, between God considered as designer of the machine of the universe, and God considered as monarch of the divine city of minds.

§2. Abstract

Let $K'(X', Y', Z')$ be a Galilei transformation of an inertia coordinate system $K(X, Y, Z)$ such that $X' = X + v_2 t$, $Y' = Y$, $Z' = Z$, where v_2 denotes the relative velocity of K' with respect to K. Assume that there is a particle B whose velocity is v_1 with respect to K'. Then G. Galilei claimed that the velocity of B with respect to K would be equal to $v_1 + v_2$.



This classical addition theorem of velocities is extremely valid whenever the values of v_1 and v_2 are sufficiently smaller than the velocity c of light. It follows from the special theory of relativity due to Einstein-Poincaré that $(v_1+v_2)(1+c^{-2}v_1v_2)^{-1}$ is the right addition theorem of velocities.

By applying the theory of formal group laws, we shall give an arithmetic proof of the above addition theorem without using light.

§3. Historical introduction

We give here a historical review of how the special relativity showed up in rough outline. Before 1881, theoretical physicists described the propagation of light through a medium, which was called the (luminiferous) ether, but there had been serious objections to this point of view. Since the earth is moving through the ether, they considered that the velocity of light in different directions should be different. This was shown to be false by the null result of the Michelson-Morley experiment in 1887.

H.A. Lorentz suggested the contraction and extended his idea to explain the failure of ether-detection experiments. Though Poincaré stated that no experiment can detect absolute motion, he did not give up the ether.

On the other hand, Einstein began with the following two axioms without gravitation ;

(i) the covariance of the physical laws ,

(ii) the constancy of the velocity of light ,

in his 1905 paper "On the electrodynamics of moving bodies".

In 1908, H. Minkowski wrote as follows. "Henceforth space by itself, and time by itself are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality."

Thus the special theory of relativity due to Einstein-Poincaré has been completed by Minkowski's revolutional change for the concept of the four-dimensional space-time.

§4. Several proofs

Let us review the following known proofs of the addition theorem of velocities, which is written briefly by ATV.

(I) Einstein proof ; In fact, he used a geometrical axiomatic method. The crucial point of his proof depends on the theory of functional equation.

(II) Poincaré proof ; In many books of the special relativity, we can find this standard proof which is using the Lorentz transformations.

(III) Minkowski proof ; Using the hyperbolic geometry of Minkowski space-time, it follows that ATV is nothing but the addition theorem of hyperbolic tangent.

(IV) Mermin proof ; He proved ATV by only one axiom of the constancy of the velocity of light. Furthermore, he obtained another proof of ATV without using light.

We would like to explain our arithmetic proof of ATV. By identifying the space of velocities to the Lie algebra of a certain kind of Dieudonné typical subgroup laws of the formal Lorentz group law $\mathbb{L}^*(3,1)$, we can calculate directly the ATV.

§5. Formal Lorentz group

Roughly speaking, a n-dimensional formal group law $\mathbb{F}=\mathbb{F}(\mathbb{X},\mathbb{Y})$ consists of a n-tuple of formal power series $F_i(\mathbb{X},\mathbb{Y})$ with $2n$ variables $\mathbb{X}=(X_1,\cdots,X_n)$, $\mathbb{Y}=(Y_1,\cdots,Y_n)$, which holds $\mathbb{F}(\mathbb{F}(\mathbb{X},\mathbb{Y}),\mathbb{Z})=\mathbb{F}(\mathbb{X},\mathbb{F}(\mathbb{Y},\mathbb{Z}))$ and $\mathbb{F}(\mathbb{X},\mathbb{O})=\mathbb{F}(\mathbb{O},\mathbb{X})=\mathbb{X}$. For an expository reference of formal groups, see [10].

Though the set-theoretical Lorentz group is the most important ingredient in the special theory of relativity, we shall introduce here six-dimensional formal Lorentz group law $\mathbb{L}^*(3,1)$ for non-commutative addition theorem of velocities. Let $\mathbb{X}=(X_1,\cdots,X_6)$ be a generic point of $\mathbb{L}^*(3,1)$, and write by $B=\begin{pmatrix} I_3 & \mathbb{O} \\ \mathbb{O} & -I_1\end{pmatrix}$, where I_n denotes the identity matrix of size n.

Then the Lie algebra $Lie(\mathbb{L}^*(3,1))$ of $\mathbb{L}^*(3,1)$ can be written as follows.

$$Lie(\mathbb{L}^*(3,1))=\left\{ \widetilde{\mathbb{X}}\in Mat(4\times4)\ ;\ {}^t\widetilde{\mathbb{X}}B+B\widetilde{\mathbb{X}}=\mathbb{O}_4 \right\}$$

$$,\text{where}\quad \widetilde{\mathbb{X}}=\begin{pmatrix} 0 & X_1 & X_2 & X_4 \\ -X_1 & 0 & X_3 & X_5 \\ -X_2 & -X_3 & 0 & X_6 \\ X_4 & X_5 & X_6 & 0 \end{pmatrix}.$$

Write by $M=(I_4+\widetilde{\mathbb{X}})(I_4-\widetilde{\mathbb{X}})^{-1}$, then ${}^tMBM$

$$=(I_4-{}^t\widetilde{\mathbb{X}})^{-1}(B+{}^t\widetilde{\mathbb{X}}B+B\widetilde{\mathbb{X}}+{}^t\widetilde{\mathbb{X}}B\widetilde{\mathbb{X}})(I_4-\widetilde{\mathbb{X}})^{-1}$$

$$=(I_4-{}^t\widetilde{\mathbb{X}})^{-1}(B-B\widetilde{\mathbb{X}}^2)(I_4-\widetilde{\mathbb{X}})^{-1}$$

$$=(I_4-{}^t\widetilde{\mathbb{X}})^{-1}B(I_4+\widetilde{\mathbb{X}})(I_4-\widetilde{\mathbb{X}})(I_4-\widetilde{\mathbb{X}})^{-1}$$

$$=\left\{(I_4+\widetilde{\mathbb{X}})^{-1}B^{-1}(I_4-{}^t\widetilde{\mathbb{X}})\right\}^{-1}=\left\{(B(I_4+\widetilde{\mathbb{X}}))^{-1}(I_4-{}^t\widetilde{\mathbb{X}})\right\}^{-1}$$

$$=\left\{(B-{}^t\widetilde{\mathbb{X}}B)^{-1}(I_4-{}^t\widetilde{\mathbb{X}})\right\}^{-1}=\left\{B^{-1}(I_4-{}^t\widetilde{\mathbb{X}})^{-1}(I_4-{}^t\widetilde{\mathbb{X}})\right\}^{-1}$$

$=B$. Hence M is a Lorentz transformation matrix.

It follows from $\mathbb{Z}=\mathbb{L}^*(3,1)(\mathbb{X},\mathbb{Y})$ that

$$\widetilde{\mathbb{Z}}=\left\{(I_4+\widetilde{\mathbb{X}})(I_4-\widetilde{\mathbb{X}})^{-1}(I_4+\widetilde{\mathbb{Y}})-(I_4-\widetilde{\mathbb{Y}})\right\}\left\{(I_4+\widetilde{\mathbb{X}})(I_4-\widetilde{\mathbb{X}})^{-1}(I_4+\widetilde{\mathbb{Y}})+(I_4-\widetilde{\mathbb{Y}})\right\}^{-1}.$$

Thus we obtain an explicit group law of $\mathbb{L}^*(3,1)$ as follows.

$$\left\{\mathbb{L}^*(3,1)(\mathbb{X},\mathbb{Y})\right\}^{-1}=(I_4-\widetilde{\mathbb{X}})^{-1}(\widetilde{\mathbb{X}}+\widetilde{\mathbb{Y}})(I_4+\widetilde{\mathbb{X}}\widetilde{\mathbb{Y}})^{-1}(I_4-\widetilde{\mathbb{X}}).$$

§6. Dieudonné typical formal subgroup laws

The similar method is available for $\mathbb{Z} = SO^*(n)(\mathbb{X}, \mathbb{Y})$, where $SO^*(n)$ denotes the special orthogonal formal group law with dimension $(n^2-n)/2$. For a generic point $\mathbb{X} = (X_1, \cdots, X_{(n^2-n)/2})$ of $SO^*(n)$, we write by $\widetilde{\mathbb{X}}$

$$= \begin{pmatrix} 0 & X_1 & X_2 & \cdot & \cdot & \cdot & X_{n-1} \\ -X_1 & 0 & X_n & \cdot & \cdot & \cdot & X_{2n-3} \\ -X_2 & -X_n & 0 & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & 0 & X_{(n^2-n)/2} \\ -X_{n-1} & -X_{2n-3} & \cdot & \cdot & \cdot & -X_{(n^2-n)/2} & 0 \end{pmatrix} \text{ in } Lie(SO^*(n)).$$

By the Cayley theorem, it is known that $(I_n+\widetilde{\mathbb{X}})(I_n-\widetilde{\mathbb{X}})^{-1}$ is an orthogonal matrix. It follows from
$(I_n+\widetilde{\mathbb{X}})(I_n-\widetilde{\mathbb{X}})^{-1}(I_n+\widetilde{\mathbb{Y}})(I_n-\widetilde{\mathbb{Y}})^{-1} = (I_n+\widetilde{\mathbb{Z}})(I_n-\widetilde{\mathbb{Z}})^{-1}$
that $\{(I_n+\widetilde{\mathbb{X}})(I_n-\widetilde{\mathbb{X}})^{-1}(I_n+\widetilde{\mathbb{Y}})-(I_n-\widetilde{\mathbb{Y}})\}(I_n-\widetilde{\mathbb{Y}})^{-1}$
$= \widetilde{\mathbb{Z}}\{I_n-\widetilde{\mathbb{Y}}+(I_n+\widetilde{\mathbb{X}})(I_n-\widetilde{\mathbb{X}})^{-1}(I_n+\widetilde{\mathbb{Y}})\}(I_n-\widetilde{\mathbb{Y}})^{-1}$.
Hence we have

$$\widetilde{\mathbb{Z}} = \{(I_n+\widetilde{\mathbb{X}})(I_n-\widetilde{\mathbb{X}})^{-1}(I_n+\widetilde{\mathbb{Y}})-(I_n-\widetilde{\mathbb{Y}})\}\{(I_n+\widetilde{\mathbb{X}})(I_n-\widetilde{\mathbb{X}})^{-1}(I_n+\widetilde{\mathbb{Y}})+(I_n-\widetilde{\mathbb{Y}})\}^{-1}.$$

Since $(I_n+\widetilde{\mathbb{X}})(I_n-\widetilde{\mathbb{X}})^{-1} = (I_n-\widetilde{\mathbb{X}})^{-1}(I_n+\widetilde{\mathbb{X}})$, one sees that

$$\widetilde{\mathbb{Z}} = (I_n-\widetilde{\mathbb{X}})^{-1}\{(I_n+\widetilde{\mathbb{X}})(I_n+\widetilde{\mathbb{Y}})-(I_n-\widetilde{\mathbb{X}})(I_n-\widetilde{\mathbb{Y}})\}\{(I_n-\widetilde{\mathbb{X}})^{-1}((I_n-\widetilde{\mathbb{X}})(I_n-\widetilde{\mathbb{Y}})+(I_n+\widetilde{\mathbb{X}})(I_n+\widetilde{\mathbb{Y}}))\}^{-1}$$

$$= (I_n - \widetilde{\mathfrak{X}})^{-1} 2 (\widetilde{\mathfrak{X}} + \widetilde{\mathfrak{Y}}) \left\{ (I_n - \widetilde{\mathfrak{X}})^{-1} 2 (I_n + \widetilde{\mathfrak{X}} \widetilde{\mathfrak{Y}}) \right\}^{-1}$$

$$= (I_n - \widetilde{\mathfrak{X}})^{-1} 2 (\widetilde{\mathfrak{X}} + \widetilde{\mathfrak{Y}}) \left\{ 2 (I_n + \widetilde{\mathfrak{X}} \widetilde{\mathfrak{Y}}) \right\}^{-1} (I_n - \widetilde{\mathfrak{X}}) .$$

Therefore we obtain that

$$\left\{ SO^*(n)(\mathfrak{X}, \mathfrak{Y}) \right\}^{\sim} = (I_n - \widetilde{\mathfrak{X}})^{-1} (\widetilde{\mathfrak{X}} + \widetilde{\mathfrak{Y}}) (I_n + \widetilde{\mathfrak{X}} \widetilde{\mathfrak{Y}})^{-1} (I_n - \widetilde{\mathfrak{X}}) .$$

In a case of $SO^*(2)$, it follows from $\widetilde{\mathfrak{X}} = \begin{pmatrix} 0 & X \\ -X & 0 \end{pmatrix}$ that

$$\left\{ SO^*(2)(X, Y) \right\}^{\sim} = \begin{pmatrix} 0 & (X+Y)(1-XY)^{-1} \\ -(X+Y)(1-XY)^{-1} & 0 \end{pmatrix} .$$

Let $\mathbb{G}_a$, $\mathbb{G}_m$ be the additive 1-dimensional formal group law, the multiplicative 1-dimensional formal group law, respectively. Furthermore we write by $F(X,Y) = (X+Y)(1+XY)^{-1}$, which is called the Lorentz boost.

Let us define the Dieudonné typical subgroup low of a n-dimensional formal group law $\mathbb{G}$. Let Λ be a non-empty subset of $\{1, 2, 3, \cdots, n\}$, and r be the number of elements of Λ. For a generic point $\mathfrak{X} = (X_1, \cdots, X_n)$ of $\mathbb{G} = \mathbb{G}(\mathfrak{X}, \mathfrak{Y}) = (G_i(\mathfrak{X}, \mathfrak{Y}))_{1 \leq i \leq n}$, we write by $\mathfrak{X}'_{(0)} = (X'_1, \cdots, X'_n)$ as follows.

$X'_j = X_j$ or 0, in accordance with $j \in \Lambda$ or $j \notin \Lambda$, respectively.

Moreover, we write by $\mathfrak{X}'=(X_j')_{j\in\Lambda}$. A r-tuple of formal power series with $2r$-variables $(G_j(\mathfrak{X}';\mathfrak{Y}'))_{j\in\Lambda}$ is called a Dieudonné typical subgroup law (of $\mathfrak{G}$) with dimension $r=\#(\Lambda)$ if and only if $G_i(\mathfrak{X}'_{(0)},\mathfrak{Y}'_{(0)})=0$ for every $i\in\Lambda$ $(1\leqq i\leqq n)$.

In fact, we can know the following table of the distribution of 1-dimensional Dieudonné typical subgroup laws of classical linear formal group laws.

	$\mathbb{G}_a$	$\mathbb{G}_m$	$SO^*(2)$	F
$GL^*(n)$	n^2-n	n	0	0
$SL^*(n)$	n^2-n	$n-1$	0	0
$SO^*(n)$	0	0	$(n^2-n)/2$	0
$Sp^*(n)$	n^2+n	0	0	n^2
$\mathbb{L}^*(3,1)$ $(=SO^*(3,1))$	0	0	3	3
$SO^*(r,n-r)$ $(1\leqq r<n)$	0	0	$(n^2-n)/2-r(n-r)$	$r(n-r)$

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Appendix

The following two references, which are well known since 1905/1906, due to A. Einstein, H. Poincaré, respectively.

[1] I. Kinematic Part
§1. Definition of simultaneity
§2. On the relativity of lengths and times
§3. Theory of transformation of coordinates and time from a system at rest to a system in uniform translational motion relative to it
§4. The physical meaning of the equations obtained concerning moving rigid bodies and moving clocks
§5. The addition theorem of velocities

II. Electrodynamic Part
§6. Transformation of the Maxwell-Hertz equations for empty space. On the nature of the electromotive forces that arise upon motion in a magnetic field
§7. Theory of Doppler's principle and of aberration
§8. Transformation of the energy of light rays. Theory of the radiation pressure exerted on perfect mirrors
§9. Transformation of the Maxwell-Hertz equations when convection currents are taken into consideration
§10. Dynamics of the (slowly accelerated) electron

[8] Table des matières

Introduction

1. Transformation de Lorentz
2. Principe de moindre action
3. La transformation de Lorentz et principe de moindre action
4. Le Groupe de Lorentz
5. Ondes de Langevin
6. Contraction des Électrons
7. Mouvement quasi stationnaire
8. Mouvement quelconque
9. Hypothèses sur la Gravitation

Here we quote the addition theorem of velocities from § 5 in [7], and from § 4 in [8] as follows.

§ 5. Additionstheorem der Geschwindigkeiten.

In dem längs der X-Achse des Systems K mit der Geschwindigkeit v bewegten System k bewege sich ein Punkt gemäß den Gleichungen:

$$\xi = w_\xi \tau, \qquad \eta = w_\eta \tau, \qquad \zeta = 0,$$

wobei w_ξ und w_η Konstanten bedeuten.

Gesucht ist die Bewegung des Punktes relativ zum System K. Führt man in die Bewegungsgleichungen des Punktes mit Hilfe der in § 3 entwickelten Transformationsgleichungen die Größen x, y, z, t ein, so erhält man:

$$x = \frac{w_\xi + v}{1 + \frac{v w_\xi}{V^2}} t,$$

$$y = \frac{\sqrt{1 - \left(\frac{v}{V}\right)^2}}{1 + \frac{v w_\xi}{V^2}} w_\eta t,$$

$$z = 0.$$

Das Gesetz vom Parallelogramm der Geschwindigkeiten gilt also nach unserer Theorie nur in erster Annäherung. Wir setzen:

$$U^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2,$$

$$w^2 = w_\xi^2 + w_\eta^2$$

und

$$\alpha = \operatorname{arctg} \frac{w_y}{w_x};$$

α ist dann als der Winkel zwischen den Geschwindigkeiten v und w anzusehen. Nach einfacher Rechnung ergibt sich:

$$U = \frac{\sqrt{(v^2 + w^2 + 2 v w \cos\alpha) - \left(\frac{v w \sin\alpha}{V}\right)^2}}{1 + \frac{v w \cos\alpha}{V^2}}.$$

Es ist bemerkenswert, daß v und w in symmetrischer Weise in den Ausdruck für die resultierende Geschwindigkeit eingehen. Hat auch w die Richtung der X-Achse (Ξ-Achse), so erhalten wir:

$$U = \frac{v + w}{1 + \frac{v w}{V^2}}.$$

Aus dieser Gleichung folgt, daß aus der Zusammensetzung zweier Geschwindigkeiten, welche kleiner sind als V, stets eine Geschwindigkeit kleiner als V

4. — Le groupe de Lorentz.

Il importe de remarquer que les transformations de Lorentz forment un groupe.

Si l'on pose en effet :

$$x' = kl(x + \varepsilon t), \quad y' = ly, \quad z' = lz, \quad t' = kl(t + \varepsilon x),$$

et, d'autre part,

$$x'' = k'l'(x' + \varepsilon' t'), \quad y'' = l'y', \quad z'' = l'z', \quad t'' = k'l'(t' + \varepsilon' x'),$$

avec

$$k^{-2} = 1 - \varepsilon^2, \quad k'^{-2} = 1 - \varepsilon'^2,$$

il viendra :

$$x'' = k''l''(x + \varepsilon'' t), \quad y'' = l''y, \quad z'' = l''z, \quad t'' = k''l''(t + \varepsilon'' x),$$

avec

$$\varepsilon'' = \frac{\varepsilon + \varepsilon'}{1 + \varepsilon\varepsilon'}, \quad l'' = ll', \quad k'' = kk'(1 + \varepsilon\varepsilon') = \frac{1}{\sqrt{1 - \varepsilon''^2}}.$$

Si nous donnons à l la valeur 1, que nous supposions ε infiniment petit,

$$x' = x + \delta x, \quad y' = y + \delta y, \quad z' = z + \delta z, \quad t' = t + \delta t,$$

il viendra :

$$\delta x = \varepsilon t, \quad \delta y = \delta z = 0, \quad \delta t = \varepsilon x.$$

C'est là la transformation infinitésimale génératrice du groupe, que