History or Heritage? A Central Question in the Historiography of Mathematics

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> However eager to tell us how scientists of the seventeenth century used their inheritance from the sixteenth, the scholars seem to regard as irrelevant anything a scientist today might think about any aspects of science, including his own debt to the past or reaction against it. C.A. Truesdell III (1968, foreword)

> You think that the world is what it looks like in fine weather at noonday; I think that it seems like in the early morning when one first wakes from deep sleep.
> A.N. Whitehead to B. Russell (Russell 1956, 41)

1. The pasts and the futures

The growth in interest and work in the history of mathematics in the last three decades or so has led naturally to reactions among mathematicians. Some of them have been welcoming, and indeed have contributed their own historical research; but many others have been cautious, and even contemptuous about the work produced by practising historians for apparently limited lack of knowledge of mathematics.[1] By the latter they usually mean the current version of the mathematics in question, and the failure of historians to take due note of it.

There is a deep distinction involved here, which has not been much discussed in the literature; even the survey (May 1976) of historiography jumps across it. I use the words 'history' and 'heritage' to name two interpretations of a mathematical theory (or definition, proof-method, algorithm or whatever); I shall use the word 'notion' as the umbrella term, and the letter 'N' to denote it. A sequence of notions in recognised order in a mathematical theory is notated 'N_0, N_1, N_2, …'.

By 'history' I refer to the details of the development of N: its pre-history and concurrent developments; the chronology of progress, as far as it can be determined (well-known to be often difficult or even impossible for ancient and also ethno-mathematics); and maybe also the impact in the immediately following years and decades. History addresses the question 'what happened in the past?'. It should also address the dual question 'what did not happen in the past?', where false starts, missed opportunities (Dyson 1972), sleepers and repeats are noted. The (near-)absence of later notions from N is registered; *differences* between N and seemingly similar more modern notions are likely to be emphasised.

By 'heritage' I refer to the impact of N upon later work, both at the time and afterwards, especially the forms which it may take, or be embodied, in modern contexts.[2] Some modern form of N is the main focus, but attention is also paid to the course of its development. Here the mathematical relationships will be noted, but not the historical ones in the above sense. Heritage addresses the question 'how did we get here?', and often the answer reads like 'the royal road to me'. The modern notion is thereby unveiled (a nice word proposed by Henk Bos); *similarities* between old and more modern notions are likely to be emphasised. In the case of sequences, a pernicious case arises when N_1 is a logical consequence or a generalisation of N_0, and the claim is made that a knower of N_0 knew N_1 also (May 1975a; an example is given in §3.4).

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[1] Another point of division between the two disciplines is techniques and practices specific to historical work, such as the finding, examination and deployment of manuscript sources and of large-scale bibliographies. (The latter are rehearsed, at least for the pre-computer age, in (May 1973, 3-41).) They are not directly relevant to this paper.

[2] In recent lectures on this topic I used the 'word' genealogy' to name this concept. I now prefer 'heritage', partly on semantic grounds and partly for its attractive similarity with 'history' in English as another three-syllable word beginning with 'h'.

Both kinds of activity are quite legitimate, and indeed important in their own right; in particular, mathematical research often seems to be conducted in a heritage-like way, although the predecessors may well be very recent (as far back as five years, say). *The confusion of the two kinds of activity is not legitimate,* either taking heritage to be history (mathematicians' common view) or taking history to be heritage (the occasional burst of over-enthusiasm by an historian): indeed, such conflations may well mess up both categories, especially the historical record.

A philosophical difference is that heritage tends to focus upon knowledge alone (theorems as such, and so on), while history also seeks causes and understanding in a more general sense. The distinction made by historians between 'internal' and 'external' history is only part of this difference. Each category is explicitly meta-theoretical, though history may demand the greater finesse in the handling of different levels of theory.

Two prominent types of writing in which heritage is the main guide are review articles and lengthy reports. Names, dates and references are given frequently, and chronology (of publication) may well be checked quite scrupulously; but motivations, cultural background, processes of genesis, and historical complications are usually left out. A golden period in report writing was at the turn of the 19th and 20th centuries, especially in German, with two main locations: the reports in the early volumes of the *Jahresberichte* of the *Deutsche Mathematiker-Vereinigung* (1892-) and the articles comprising the *Encyklopädie der mathematischen Wissenschaften* (1898-1935) with its unfinished extension into the French *Encyclopédie des sciences mathématiques* (1904-1920?) (Gispert 1999). The difference between history and heritage was not always strong at that time;[3] for example, a few of the *Encyklopädie* reports are quite historical.

Among modern examples of heritage-oriented writings, Jean Dieudonné's lengthy account of algebraic and differential topology in the 20th century is typical (Dieudonné 1989), and several of the essays in the Bourbaki history have the same character (Bourbaki 1974). André Weil's widely read advice (1980) on working on history is driven more by needs of heritage, especially concerning judgements of importance; but it is somewhat more nuanced in other respects. An interesting slip is his use of 'history of mathematics' and 'mathematical history' as synonyms, whereas they are quite different subjects (Grattan-Guinness 1997, 759-761).

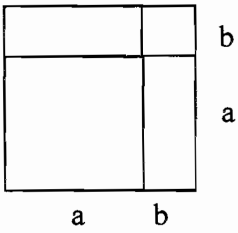
A third category arises when N is laid out completely time-free with all developments omitted, historical or otherwise; for example, as a strictly axiomatised theory. This kind of writing is also quite legitimate, but is neither history nor heritage (though it may *have* both), and I shall not consider it further.

[3] See (Dauben 1999) on the journals for the history of mathematics then.

2. An example

This distinction has been cast in as general a manner as possible; any piece of mathematics from any culture will be susceptible to it. Here is an example, mathematically simple but historically very important (this last remark itself a manifestation of the distinction from heritage, note).

In his *Elements* Euclid gives this theorem about 'completing the square':



The historical interpretation of Euclid as a closet algebraist developed during the late 19th century (compare the remarks in §1 on history and heritage at that time); thus the diagram has long been rendered in algebraic form as

$$(a + b)^2 = a^2 + 2ab + b^2. \qquad (1)$$

However, mathematical as well as historical disquiet should arise. Firstly, (1) is a piece of algebra, which Euclid did not use, even covertly: his diagram does not carry the letters 'a' and 'b'. His theorem concerned geometry, about the large square being composed of four parts, with rectangles to the right and above the smaller square and a little square off in the north-east corner. But these geometrical relationships, essential to the theorem, are lost in the single sign '+'. Further, 'a' and 'b' are associated with numbers, and thereby with lengths and their multiplication. But Euclid worked with lines, regions, solids and angles, not any arithmeticised analogues such as lengths, areas, volumes and degrees; he never multiplied geometrical magnitudes of any kind (though multiplication of numbers in arithmetic was practised). Hence 'a^2' is already a distortion; he constructed the 'square *on* the side', not the 'square *of* the side' (Grattan-Guinness 1996). For reasons such as this the algebraic reading of Euclid has been discredited in recent decades by specialists; by contrast, it is still advocated by mathematicians, such as (Weil 1980) who even claims that group theory is *needed* in order to understand Books 5 and 7 of Euclid!!

These are historical and meta-historical remarks about Euclid; (1) belongs to its heritage, especially among the Arabs with their word-based algebra (the phrase 'completing the square' is Arabic in origin), and then in European mathematics, with symbols for quantities and operations gradually being introduced.[4] The actual version (1) corresponds to the early 17th century, with figures such as Thomas Harriot and René Descartes; Euclid and the Arabs are part of their history,

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[4] There is of course another large history and heritage from Euclid, inspired by the alleged rigour of this proofs. It links in part to the modernisation of his geometry, but I shall not discuss them here.

they are part of the heritage from Euclid and the Arabs, and *our* use of (1) forms part of our heritage from them.[5]

3. Some attendant distinctions

The distinction between history and the heritage of N seems to be that between its relationship to its pre-history and to its post-history. If N_0, N_1 and N_2 lie in advancing chronological order, then the heritage of N_1 for N_2 belongs also to the history of N_2 relative to N_0 and N_1. However, the situation is not so simple; in particular, both categories use the post-history of N, though in quite different ways. Thus more needs to be discussed. Some further examples will be used below, though for reasons of space and balance rather briefly; fuller historical accounts would take note of interactions of the development of other relevant notions.

3.1 *History is usually a story of heritages.* The historian records events where normally an historical figure inherited knowledge from the past in order to make his own contributions. If the figure really did treat a predecessor in an historical spirit (as he (mis-)understood it), then the (now meta-)historian should record accordingly (for example, (Stedall 2001) on John Wallis's *Algebra* of 1685).

3.2 *Types of influence* raise important issues. However, research is likely to focus only upon positive influence whereas history needs to take note also of negative influences, especially of a general kind, such as reaction against some notion or the practise of it or importance accorded some context. For example, one motive of A.-L. Cauchy to found mathematical analysis in the 1820s upon a theory of limits (§4.1) was his rejection of J.L. Lagrange's approach to the calculus using only notions from algebra. Further, as part of his new regime Cauchy stipulated that 'a divergent series has no sum'; but in the 1890s Emile Borel reacted against precisely this decree and became a major figure in the development of summability and formal power series (Tucciarone 1973). Part of the heritage of those theories has been to treat as idiots pre-Cauchyesque manipulators of infinite series such as Leonhard Euler!

3.3 *The role of chronology* differs greatly. In history it can form a major issue; for example, possible differences between the creations of a sequence of notions and those of their publication. Further, the details available may only give a crude or inexact time course, and some questions of chronology remain unanswerable. In heritage chronology is much less significant, apart from questions of the type 'Who was the first mathematician to ...?'. Mathematicians often regard them as the prime type of historical question to pose (May 1975b), whereas historians recognise them as often close to meaninglessness when the notion involved is very general or

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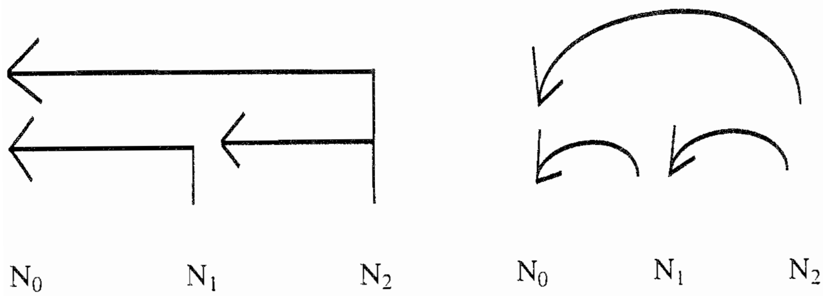
[5] This last feature applies also, regrettably, to the supposed history (Rashed 1994) of Arabic algebra, where the Arabs seem already to have read Descartes.

basic; for example, '... to use a function?' could excite a large collection of candidates according to the state, generality or abstractness of the function theory involved. The only type of questions of this kind of genuine historical interest concerns priority disputes, when intense parallel developments among rivals are under investigation, and chronology is tight — and where again maybe no answer can be found.

3.4 *Uses of later notions.* They are *not* to be ignored; the idea of forgetting the later past of an historical episode is impossible to achieve, and indeed not desirable. Instead its status *as* later work is duly recognised, and tiers of history exposed: work produced in, say, 1701 was historical in 1801 and in 1901 as well as now in 2001. Thus, when studying the history of N_0, recognise the place of later notions N_1, N_2,... but *avoid* feeding them back into N_0 itself. For if that does happen, the novelties that attended the emergence of N_1, N_2,... will not be registered. Instead time loops are created, with cause and effect over time becoming reversed: when N_2 and N_1 are shoved into N_0, then they seem to be involved in its creation, whereas the *converse* is (or may be) the case. In such situations not only is the history of N messed up but also that of the intruding successors, since their *absence* before introduction is not registered. For example, Lagrange's work in algebra played a role in the certain aspects of group theory (Wussing 1984, 70-84); but to describe his work in terms of group theory not only distorts Lagrange but also muddies the (later) emergence of group theory itself. By contrast, the heritage may be clarified by such procedures, and chaos in the resulting history is not significant.

A valuable use of later notions when studying the history of N is as a source for questions to ask about N itself — but do not expect positive answers! (The converse may well hold; knowing at least some of the history of N_0, N_1, N_2, ... may well increase understanding of their relations, and even suggest a research topic.) By contrast, when studying the heritage of N_0, by all means feed back N_1, N_2 ... to create new versions and with luck find a topic for mathematical research. The difference is shown below; for history the horizontal arrows do not impinge positively upon the preceding notions whereas those for heritage do:



History Heritage

The difference is often exemplified by reactions to older mathematics. The inheritor reads something by, say, Lagrange and exclaims: 'My word, Lagrange here is very modern!'. The historian replies: 'No, we are very Lagrangian'

The distinction between history and heritage is thus emphatically *not* that between success and failure; history also records successes, but with the slips and delays exposed. For example, A nice example is (Hawkins 1970), a fine history of the application of point set topology to refine the integral from the Cauchy-Riemann version through content in the sense of Jordan and Cantor to the measure theory of Henri Lebesgue and Borel. Hawkins not only records the progress achieved but also carefully recounts conceptual slips made en route: for example, the belief until its exposure that denumerable set, set of measure zero and nowhere dense set were co-extensive concepts.

3.5 *Foundations up or down?* This distinction can be extended when N is an axiomatised theory, which proceeds logically through concepts C_1, C_2, C_3; for to some extent the respective historical origins move *backwards* in time, thus broadly the reverse of the historical record. A related difference is thereby exposed: heritage suggests that the foundations of a mathematical theory are laid down as the platform upon which it is built, whereas history shows that foundations are dug down, and nor necessarily on firm territory. For example, the foundations of arithmetic may start with mathematical logic in a version of the 1900s, use set theory as established around the 1890s, define progressions via the Peano axioms of the later 1880s, and then lay out the main properties of integers as established long before that.

A figure important in that story is Richard Dedekind, with his book of 1888 on the foundations of arithmetic. The danger of making historical nonsense out of heritage is well shown in a supposed new translation. A typical example of the text is the following passage, where Dedekind's statement that (in literal translation) 'All simply infinite systems are similar to the number-series *N* and consequently by (33) also to one another' comes out as '*All unary spaces are bijective* [1] *to the unary space* [2] *N and consequently, by §33,* [3] *also to one another* '; moreover, of the three editorial notes, the first one admits that 'isomorphic' would be more appropriate for Dedekind but the second one informs that '*unary space* [...] is what he means' ... ('Dedekind' 1995, 63).

3.6 *Indeterminism or determinism?* Especially if the history properly notes missed opportunities, delayed and late arrivals of conception and/or publication, an indeterministic character is conveyed: the history did indeed pass through the sequence of notions N_0, N_1, N_2, ..., but it might have been otherwise (unintended consequences, and so on). By contrast, even if not explicitly stressed, a deterministic impression is likely to be conveyed by heritage: N_0 *had* to lead to N_1, and so on. Appraisal of historical figures as 'progressive' or 'mordents', in any context, is normally of this kind: the appropriate features of their work are stressed, the others ignored (for example, Newton the modern scientist yes, Newton the major alchemist no).

A fine example of indeterminism id provided by the death of Bernhard Riemann in 1866. The world lost a very great mathematician, and early; on the other hand, his friend Dedekind published soon afterwards two manuscripts which Riemann had prepared in 1854 for his *Habilitation* but had left them unpublished, seemingly indefinitely. One essay dealt with the foundations of geometry, the other with mathematical analysis and especially Fourier series. Each of them made a rapid and considerable impact, and each contained notions and connections which were current in some other authors; however, if the essay on analysis had not appeared, there is no reason to assume that Georg Cantor (1845-1918), then a young number theorist, would have tackled the hitherto unnoticed problem of exceptional sets for Fourier series (to use the later name) and thereby invented the first elements of his set theory (Dauben 1979, chs. 1-2). But then many parts of mathematical analysis would have developed differently. (The bearing of the other essay on the development of geometries is noted in §3.7.) Other early deaths suggest possibilities: Evariste Galois stopping a bullet in 1832, Jacques Herbrand falling down a mountain a century later, and so on.

3.7 *Revolutions or convolutions?* When appraising heritage, interest lies mainly in the outcomes without special concern about the dynamics of their production. A deterministically construed heritage conveys the impression that the apparently inevitable progress makes mathematics a *cumulative* discipline.

History suggests otherwise; some theories die away, or at least die down in status. The status or even occurrence of revolutions in mathematics is historically quite controversial (Gillies 1992); I have proposed the meta-notion of convolution, where new and old notions wind around each other as a (partly) new theory is created (Grattan-Guinness 1992). Convolution lies between, and can mix, three standard categories: revolution, in the sense of strict *replacement* of theory; innovation, where replacement is absent or plays a minor role (I do not know of a case where even a remarkably novel notion came from literally *no* predecessors); and evolution, similar to convolution in itself but carrying many specific connotations in the life sciences which are not necessarily useful here.

One of the most common ways in which old and new mix is when a new notion is created by connecting two or more old notions in a novel way. Among very many cases, in 1593 François Viète connected Archimedes's algorithmic exhaustion of the circle using the square, regular octagon, ... with the trigonometry of the associated angles and obtained this beautiful infinite product

$$2/\pi \quad = \quad \sqrt{1/2}\sqrt{1/2 + 1/2\sqrt{\sqrt{1/2}}}\sqrt{1/2 + 1/2\sqrt{1/2 + 1/2\sqrt{1/2}}}\ \sqrt{\;}\ldots. \qquad (2)$$

Again, in the 1820s Niels Henrik Abel and Carl Jacobi independently linked the notion of the inverse of a mathematical function with Adrien-Marie Legendre's theory of 'elliptic functions' to produce their definitive theories of elliptic functions. Heritage may also lead to such connections being effected.

Sometimes convolutions, revolutions and traditions can be evident together. A very nice case is found in the work of Joseph Fourier in the 1800s on heat diffusion (Grattan-Guinness and Ravetz 1972). 1) Apart from a very unclear and limited anticipation by J.-B. Biot, he innovated the differential equation to represent the phenomenon. 2) The method that he used to obtain it was traditional, namely Euler's version of the Leibnizian differential and integral calculus (which is noted in §4.1). 3) He refined the use of boundary conditions to adjoin to the internal diffusion equation for solid bodies. 4) He revolutionised understanding of the solution of the diffusion equation for finite bodies by trigonometric series, which had been known before him but with important misunderstandings, especially about the manner in which a periodic series could represent a general function at all. 5) He innovated the Fourier integral solution, for infinite bodies.

Delays often arise from connections *not* being made. A well-known puzzle is the slowness to recognise non-Euclidean geometries when there was a long history of map-making which surely exhibits one kind of such a geometry. J.H. Lambert is an especially striking figure here, as he worked with some lustre in both areas in the later 18th century. The answer seems to be that, like his predecessors and several successors, he understood the geometry problem as being just the status, especially provability, of the parallel axiom *within the Euclidean framework* rather than the more general issue of alternative geometr*ies*, which was fully grasped only by Riemann in his 1854 essay (Gray 1989). Thus the link, which seems so clear in our heritage, was not obvious in the earlier times.

3.8 *Description or explanation?* Both history and heritage are concerned with description; but history should also attempt explanations of the developments found, and also of the delays and missed opportunities that are noticed. These explanations can be of various kinds; not just of the technical insights that were gained but also the social background, such as the (lack of) educational opportunities for mathematics in the community or country involved.

One feature especially of the 19th century which needs explanation is the differences between nations of the *(un)popularity* of topics or branches of mathematics (France doing loads of mathematical analysis, England and Ireland with rather little of it but working hard at several new algebras, and so on). Heritage studies will need to consider explanation only from a formal or epistemological point of view; for example, explaining the mystery of having to use complex numbers when finding the real roots of polynomials with real coefficients in terms of closure of operations over sets, an insight which has its own history.

3.9 *Levels of (un)importance.* This last task relates to another difference; that a notion rises and/or falls in importance. Heritage does not need to give such changes much attention; the modern level of reputation is taken for granted. But history should watch and ponder upon the changes carefully. For example, for a long time trigonometry has been an obviously useful but rather minor topic in a course in algebra — and there has been no detailed general history of it since (von Braunmühl 1900, 1903). By contrast, in the late Middle Ages it was a major branch of mathematics, handled geometrically (for example, the sine was a length, not a ratio), and with the spherical part more important than the planar (because of its use in astronomy and navigation). Conversely, probability theory and especially mathematical statistics had a very long and slow geneses; most of its principal notions in statistics are less than then two centuries old, and the cluster of them which are associated with Karl Pearson and his school has celebrated their centenary only recently. The slowness of the arrival of this discipline, now one of the most massive part of mathematics while often functioning separate from it, is one of the great mysteries of the history of mathematics; its unimportance during most of the 19th century is especially astonishing. But such features need not disturb a seeker of heritage.

3.10 *Handling muddles.* One way in which knowledge of all kinds, and especially the mathematical, increases is by the cleaning up of unclarities and ambiguities by, for example, bringing in new distinctions. Such housework forms part of the heritage which the mathematician will deploy (unless he has reason to question it). The historian will also the modern presence of such distinctions, but he should try to *reconstruct* the old unclarities, as clearly as possible, so that the history of the distinctions is itself studied (§4.1 has an important example).

3.11 *History as meta-theory.* This paper, especially in this section, carries a feature which needs emphasis: that when the historian studies his historical figures he is thinking *about* them, not *with* them. The distinction between theory and meta-theory, and especially the recognition of its *central* importance for knowledge, emerged during the 1930s principally from the logicians Kurt Gödel (1906-1978) and Alfred Tarski (1902-1983), after many partial hits and misses (Grattan-Guinness 2000, chs. 8-9).

In logic the distinction is very subtle; for example, 'and' feature in both logic and meta-logic, and failure to register it led to much inherence and even paradoxes such as 'this proposition is false'. In most other areas of thought the distinction seems to be too obvious to require emphasis; clearly a difference of category exists between, say, properties of light and laws of optics, or between a move in chess and a rule of chess. But when registered its importance can be seen, because it is *quite general.* This was the case with Tarski's theory of truth (his own main way to the distinction): 'snow is white' (in the metalanguage) if and only if snow is white (in the language). His theory is neutral with respect to most philosophies, and side-steps generations of philosophical anxiety about making true (or false) judgements or holding such beliefs.

In historiography the distinction stresses two different levels of both knowledge and of ignorance, with further levels required when intermediate historical stages are considered. It also side-steps chatter about narratives and discourses, and the relativism and determinism that often accompanies them.

3.12 *Consequences for mathematics education.* The issue of heuristics on mathematics, and the discovery and later justification of mathematical notions, are strongly present is this discussion, with obvious bearing upon mathematics education. The tradition there, especially at university level or equivalent, is to teach a mathematical theory in a manner very much guided by heritage. But reactions of students (including myself, as I still vividly recall) is often distaste and bewilderment; not particularly that mathematics is very hard to understand and even to learn but mainly that it turns up in "perfect" dried-out forms, so that if there are any mistakes, then necessarily I made them. Mathematical theories come over as all answers but no questions, all solutions but no problems. A significant part of the growth in interest in the history of mathematics has been inspired as a negative influence of such situations, and there is now a strong international movement for making use of history in the teaching of mathematics, at all levels. I have proposed the meta-theoretical notion of 'history-satire', where the historical record is respected but many of the complications of the normally messy historical record are omitted or elided (Grattan-Guinness 1973). (If one stays with, say, Newton all the time, then one will stop where Newton stopped.) Otto Toeplitz's 'genetic approach' to the calculus is close to a special case (Toeplitz 1963).

4. Six prevalent aspects

I conclude with five special cases of aspects of mathematics where the conflation of history and heritage seems to be especially serious, including among historians. They come mostly from the 19th and early 20th centuries, which not accidentally is my own main period of research; thus no claim of optimal importance or variety is made for them. Examples of the distinctions made in §3 are also included.

4.1 *The calculus and the theory of limits.* There have been four main ways of developing the calculus (Grattan-Guinness 1987): in chronological order,

1) Isaac Newton's 'fluxions' and 'fluents' (1660s onwards), with the theory or limits deployed, though not convincingly;

2) G.W. Leibniz's 'differential' and 'integral' calculus, based upon dx and ∫x (1670s onwards), with infinitesimals central to and limits absent from all the basic concepts: reformulated by Euler in the mid 1750s by adding in the 'differential coefficient', the forerunner of the derivative;

3) Lagrange's algebraisation of the theory, in an attempt to avoid both limits and infinitesimals, with a new basis sought in Taylor's power-series expansion (1770s onwards), and the successive differential coefficients reconceived in terms of the coefficients of the series as the 'derived functions'; and

4) Cauchy's approach based upon with a firm *theory* (and not just intuition) of limits (1810s onwards); from it he defined the basic notions of the calculus (including the derivative as

the limiting value of the difference quotient) and also of the theories of functions and of infinite series, to create 'mathematical analysis'.

Gradually the last tradition gained wide acceptance, with major refinements brought in with Karl Weierstrass and followers from the mid century onwards, especially the consequences of refining Cauchy's basically single-limit theory into that of multiple limits with a plethora of fine distinctions. Thus it has long been the standard way of teaching the calculus; but historians should beware using it to rewrite the history of the calculus where any of the other three traditions, even Newton's, are being studied. It also contains an internal danger. The (post-)Weierstrassian refinements have become standard fare, and are incorporated into the heritage of Cauchy; but it is mere feedback "history" to read Cauchy (and contemporaries such as Bernard Bolzano) as if they had read Weierstrass already (Freudenthal 1971). On the contrary, their own pre-Weierstrassian muddles need reconstruction, and clearly.

Again by contrast, heritage can acknowledge such anachronisms but ignore them as long as the mathematics produced is interesting.

4.2 *Part-whole theory and set theory.* An important part of Weierstrass's refinement of Cauchy's tradition was the introduction from the early 1870s of set theory, principally by Georg Cantor. Gradually it too gained a prominent place in mathematics and then in mathematics education; so again conflations lurk around its history. They can occur not only in putting set-theoretical notions into the pre-history, but in particular confusing that theory with the traditional way of handling collections from antiquity: namely, the theory of whole and parts, where a class of objects contains only parts (such as the class of European men as a part of the class of men), and membership was not distinguished from inclusion. Relative to set theory parthood corresponds to improper inclusion, but the theory can differ philosophically from Cantor's doctrine, on matters such as the status of the empty class/set, and the class/set as one and as many; so care is needed. An interesting example occurs in avoiding the algebraisation of Euclid mentioned in §2: (Mueller 1981) proposed an algebra alternative to that in (1), but he deployed set theory in it whereas Euclid had followed the traditional theory, so that a different distortion arises. As in earlier points, study focused upon heritage need feel no discomfort.

4.3 *Vectors and matrices.* In a somewhat disjointed way vector and matrix algebras and vector analysis gradually developed during the 19th century, and slowly became staple techniques during the 20th century, including in mathematics education (Grattan-Guinness 1994, articles 6.2, 6.7, 6.8, 7.12). But then the danger just highlighted arises again; for earlier work was not thought out that way. The issue is *not* just one of notation; the key lies in the associated notions, especially the concept of laying out a vector as a row or column of quantities and a matrix as a square or rectangular array, and manipulating them separately or together according to stipulated rules and definitions.

A particularly influential example of these anachronisms is Truesdell; in very important pioneering historical work of the 1950s he expounded achievements by especially Euler in continuum mathematics which previously had been largely ignored (see, for example, Truesdell 1954). However, in the spirit of heritage in his remark quoted at the head of this paper, he treated Euler as already familiar with vector analysis and some matrix theory (and also using derivatives as defined via the theory of limits whereas Euler had actually used his own elaboration of Leibniz's version of the calculus mentioned in §4.1). Therefore his Euler was out of chronological line by at least a century. It is quite amusing to read his introductory commentary and then the original texts in the same volume (11 and 12 of the second series of Euler's *Opera omnia*). A lot of historical reworking of Euler is needed, not only to clarify what and how he had actually done but also to eliminate the mess-ups of feedback and clarify the history of vectors and matrices by noting their absence in Euler.

4.4 *The status of applied mathematics.* During the middle of the 19th century the professionalisation of mathematics increased quite notably in Europe; many more universities and other institutions of higher education were created or expanded, so that the number of jobs increased. During that period, especially in the German states and then Germany, a rather snobbish preference for pure over applied or even applicable mathematics began to emerge, there and later internationally. Again this change has affected mathematics education (for the worse); it has also influenced historical work in that the history of pur(ish) topics have been studied far more than that of applications. The history of military mathematics is especially ignored.

An error concerning levels of importance arises here; for prior to the change, however, applications and applicability were very much the governing motivation for mathematics, and the balance of historical research should better reflect it. Euler is a very good case; studies of his contributions to purish mathematics far exceed those of his applied mathematics (hence the importance of Truesdell's initiative in looking in detail at his mechanics). Some negative influence from current practise is required of historians to correct this imbalance.

4.5 *The place of axiomatisation.* From the late 19th century onwards David Hilbert encouraged the axiomatisation of mathematical theories, in order to make clearer the assumptions made and also to study meta-properties of consistency, completeness and independence. His advocacy, supported by various followers, has given axiomatisation a high status in mathematics, and thence to mathematics education. But once again dangers of distortion of earlier work attend, for Hilbert's initiative was then part of a *new* level of concern with axiomatisation (Cavaillès 1938); earlier work was rarely so preoccupied, although the desire to make clear basic assumptions was frequently evident (for example, in the calculus as reviewed in §4.1). Apart from Euclid, it is seriously out of time to regard as axiomatisers any of the other figures named above, even Lagrange, Cauchy or Cantor.

4.6 *Words of general import.* One aim of many mathematical theories is generality; and attendant to this aspiration is the use of correspondingly wide-ranging words or phrases, such as 'arbitrary' or 'in any manner', to characterise notions. These expressions are still used in many modern contexts; so again the dangers of identification with their past manifestations need to be watched. A good example is the phrase 'any function' in the calculus and the related theory of functions; it or a cognate will be found with John Bernoulli in the early 18th century, Euler about 40 years later, S.-F. Lacroix around 1800, J.P.G. Dirichlet in the late 1820s, and Lebesgue and the French school of analysts in the early 20th century. Nowadays it is usually taken to refer to a mapping (maybe with special conditions such as isomorphism), with set theory used to specify range and domain and nothing else. But the universe of functions has not always been so vast; generality has always belonged to its period of assertion. In particular, (Dirichlet 1829) mentioned the characteristic function of the irrational numbers (to use the modern name); but he quite clearly regarded it as a pathological case, for it did not possess an integral. The difference is great between his situation and that of Lebesgue's time, when the integrability of such a function was a good test case of the new theory of measure to which he was a major contributor; indeed, this detail is part of the heritage from Dirichlet.

5. Concluding remark

It would be appropriate to end on the theme of generality, namely that of the distinction outlined in this paper. As was indicated in §1, it is applicable to history of any kind, especially the history of other sciences, although its prominence and importance in mathematics is rather special. Another related topic is the history of mathematics itself, where the (meta-)history of the subject needs to be distinguished from the heritage which we historians today enjoy from our predecessors (Dauben and Scriba 2002) — for example, the history of changing views on Euclid.

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