Encounter with Uniform Continuity: Cauchy's algebraic Approach vs. Dirichlet's geometrical Approach

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1.Introduction

In university courses of mathematics, professors often insist on the rigorous aspect of mathematics and introduce strange definitions that are quite different from high school style definitions. For example, they argue that it is crucial for the rigorous discussion in calculus to define continuity in terms of ε - δ inequalities: $f(x)$ is said to be continuous at $x_o \in I$ if for every ε, there exists a $\delta(\varepsilon)$ such that $|f(x) - f(x_0)| < \varepsilon$ for all points $x \in I$ for which $|x - x_o| < \delta$. If f is continuous at every point of I, then f is said to be continuous on I. Then, they develop the calculus theory based on this definition and introduce the notion of uniform continuity: $f(x)$ is said to be uniformly continuous on I if for every ε, there exists δ such that $|f(x_1) - f(x_2)| < \varepsilon$ for all x_1 and x_2 in I for which $|x - x_o| < \delta$.

By giving the definition of uniform continuity professors seem to succeed in showing the utility of the epsilontics definition, since it is quite difficult to describe a notion of δ independent of x if they adopt the same definition of continuity as that of high school mathematics. But students, exactly speaking at least when I was a student, feel somewhat unsatisfied while they realize the utility of epsilontics, because a new question arises in their mind. The notion of uniform continuity is so subtle that they don't understand why mathematicians can distinguish it from "mere" continuity. Hence, I examined how the nineteenth century mathematicians attained this notion. If I use Professor Grattan-Guinness's terminology, my motivation sounds rather "heritage" than "historical". But, I take a historical examination. I think historical discussions often help us to understand the meanings of difficult notions in mathematics. Furthermore, this examination exemplifies how the so-called rigorous theory was developed.

It is well-known that Heinrich Eduard Heine demonstrated a definition of uniform continuity distinguishing it from "mere" continuity in 1872. In the process of establishing this definition, we have a remarkable description given by Peter Gustav Lejeune-Dirichlet in 1854: "if, $y = f(x)$ is a continuous function of x in the finite interval from a to b, and by subinterval we mean difference of any two values of x, that is to say, every part of abscissa a and b. Then, it is always possible that for any chosen small positive quantity ρ, a second quantity σ, proportional to ρ, which has the following property will be found: in each subinterval which is shorter than σ, the function y varies at most by ρ." In this statement that he himself named as a fundamental property, we find that he attained a notion of uniform continuity.

Some historians of mathematics try to find a notion of uniform continuity in Augustin Louis Cauchy's and André Marie Ampère's works published in 1821 and 1824 [Bottazzini, 1990, LXXXII-LXXXV]. But, I think Dirichlet's description was the first influential demonstration of this notion. Differing from the former mathematicians, Dirichlet actually gave an example of a continuous, but in fact a non-uniformly continuous function of $y = \sin(x^2)$ in his discussion. This fact indicates that he realized that uniform continuity

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is a special case of continuity, while the former mathematicians didn't distinguish these two notions. Thus, an examination of Dirichlet's argument is indispensable in discussing the process of obtaining the notion of uniform continuity. This paper focuses on Dirichlet's remarkable statement and examines how he attained it. Since the establishment of this notion should be discussed from various aspects, this paper can submit only one possibility. However, my argument would help to demonstrate a whole historical development of this notion that has not been analyzed yet.

2. Cauchy's theory of definite integral

2-1 Epsilontics in Cauchy and Dirichlet

As modern mathematical knowledge suggests, we need an idea of ε - δ inequalities in discussing our problem related to uniform continuity. Therefore, let's confirm Cauchy's and Dirichlet's usage of these inequalities before beginning our examination. Today some of us believe that Cauchy began his analysis with defining continuity in terms of ε - δ inequalities. This is false. He used the terms "indefinitely decreases" and "indefinitely approaches 0", the so-called limit concept, to construct his theory. For example, he defined a continuous function not by ε - δ inequalities but using the former notions. But, he actually showed that a dynamical notion of the limit concept can be described in terms of inequalities. On the other hand, he clearly defined infinitesimals as a variable that indefinitely approaches 0, namely he defined them based on his limit concept. Therefore, ε - δ inequalities and infinitesimals coexisted in his framework and the limit concept could be represented in two ways: using ε - δ inequalities or infinitesimals. He seemed to select one of them or both of them for his problems. For example, to problems involving inequalities he adopted ε - δ inequalities. Compared with Cauchy, Dirichlet preferred inequalities. But, Dirichlet took almost the same attitude as Cauchy.

2-2 Cauchy's definition in *Résumé*

Dirichlet's argument appeared in his lectures on the definite integral. Since, his lectures were based on Cauchy's definition of the definite integral, we firstly examine Cauchy's integral theory as demonstrated in the *Résumé sur le calcul infinitesimal* in 1823. While, the eighteenth century mathematicians, Euler, the Bernouillis, Lagrange and Laplace, had preferred to think integration as the inverse of differentiation, Cauchy defined it as the sum of infinite series. Since, Cauchy noted that one was naturally led by "the theory of quadratures" to consider the integral as a sum in the postscript of his other paper on integral in 1823, he realized that his definite integral as a way of approximating the area that is bounded by $x = x_0$, $x = X$, $y = 0$ and the graph of $y = f(x)$.[2] However, Cauchy didn't prove his sum becomes the rectilinear approximation to curvilinear areas in the *Résumé*. He constructed his theory under the idea of algebraic analysis, namely separating geometrical image from analysis and consistently discussing it in an algebraic way .

To define his definite integral, he took a function $f(x)$ to be continuous on a given interval with end points x_0, X. He divided the interval into n not necessarily equal parts

[2] See Cauchy [1823b]. This postscript is found on pp.354-357.

$x_1 - x_0, x_2 - x_1, \cdots X - x_{n-1}$ and defined the sum

$$S = (x_1 - x_0)f(x_0) + (x_2 - x_1)f(x_1) + \cdots + (X - x_{n-1})f(x_{n-1}). \quad (1)$$

It is Cauchy's great insight not to finish his definition by only mentioning that the definite integral was the limit sum. He noted that it was necessary to prove that S has a unique limit when the size of the subintervals becomes very small and n very large, because the value of S depends on both n and the mode of division of the interval.

To prove this fact Cauchy at first regarded that $[x_0, X]$ was constructed by only one interval. Then, he noted that if one adopted the idea of mean value, the right hand of equation (1) should be replaced by $(X - x_0)f[x_0 + (X - x_0)]$ for some constant θ between 0 and 1. Therefore, he applied the same technique to each of the n subintervals $x_1 - x_0$, $x_2 - x_1$, $\cdots$, $X - x_{n-1}$ and obtained

$$\begin{aligned} S' &= (x_1 - x_0)f[x_0 + \theta_0(x_1 - x_0)] + (x_2 - x_1)f[x_1 + \theta_1(x_2 - x_1)] \\ &\quad + \cdots + (X - x_{n-1})f[x_{n-1} + \theta_{n-1}(X - x_{n-1})] \quad (2) \end{aligned}$$

where θ_i is between 0 and 1. Then he defined a set of ε_k values for which $k = 0, 1, \cdots, n-1$, by

$$f[x_k + \theta_k(x_{k+1} - x_k)] = f(x_k) \pm \varepsilon_k \quad (3)$$

so that

$$\begin{aligned} S' &= (x_1 - x_0)[f(x_0) \pm \varepsilon_0] + (x_2 - x_1)[f(x_1) \pm \varepsilon_1] + \cdots + (X - x_{n-1})[f(x_{n-1}) \pm \varepsilon_{n-1}] \\ &= S + D \quad (4) \end{aligned}$$

where, $D = \pm\varepsilon_0(x_1 - x_0) \pm \varepsilon_1(x_2 - x_1) + \cdots \pm \varepsilon_{n-1}(X - x_{n-1})$. Cauchy argued that if the length of the subinterval $x_k - x_{k-1}$ are taken sufficiently small, then the ε_k will become very close to zero, therefore $D = \varepsilon(X - x_0)$, where ε is a mean of the $\pm\varepsilon_k$. Then, when taking a subportion of the original portions it will not appreciably change the value of S. Cauchy continued to do further examination and finally succeeded in showing that the value of S reaches a certain limit, that depends only on $f(x)$ and the end point of x_0 and X of the interval. He named this limit as a definite integral and wrote $\int_{x_0}^{X} f(x)dx$, etc in the 21^{st} lesson of the *Résumé*.

Modern readers can easily find a theoretical weakness in Cauchy's procedure; his argument about D approaching to 0 is valid for uniformly continuous functions, though he assumed the "mere" continuous functions in his integral theory. But, we also note this problem is not crucial for his discussion because we know that any continuous function on a closed interval, the case that Cauchy dealt with, becomes uniformly continuous. Therefore, no theoretical contradiction could occur if he had restricted his theory to closed intervals.

However, Cauchy dealt with the case where x_0 or X become infinite, or $f(x)$ became unbounded at x_0 or X of the 24^{th} lesson in *Résumé*. Cauchy noted that the definite integral could be decomposed as

$$\begin{aligned} \int_{x_0}^{X} f(x)dx &= \int_{x_0}^{\xi_0} f(x)dx + \int_{\xi_0}^{\xi} f(x)dx + \int_{\xi}^{X} f(x)dx \\ &= (\xi_0 - x_0)f[x_0 + \theta_0(\xi_0 - x_0)] + \int_{\xi_0}^{\xi} f(x)dx + (\xi - x)f[\xi + \theta(X - \xi)] \quad (5) \end{aligned}$$

where, $0 < \theta_0,\ \theta < 1$ and if ξ_0 is very close to $x = 0$ and X to ξ_0. Hence, he obtained

$$\int_{x_0}^{X} f(x)dx = \lim \int_{\xi_0}^{\xi} f(x)dx \tag{6}$$

when ξ_0 converges to x_0 and ξ to X. Using this equation he calculated several integrals, including $\int_{-\infty}^{0} e^x ds = 1$, $\int_0^1 \frac{dx}{x} = \log 1 - \log 0 = \infty$. That is to say that he generalized his discussion of continuous functions on closed intervals for the ones with open intervals.

Cauchy's generalization directs our attention to his argument for D approaching 0. He implicitly used it when he defined $\int_{\xi_0}^{\xi} f(x)dx$, which appeared in the second term of the right hand side of the equation (5). Thus, he correctly used the argument, because he applied it only to the continuous function on the closed interval, namely an uniformly continuous function. Sometimes modern mathematicians blame Cauchy that he didn't describe his theory in terms of ε - δ inequalities. If he had done so, he should have realized his weakness, which arose from the indifference of uniform continuity.[3] But, he had no reason to change his approach because he didn't encounter with a situation that would induce the weakness which caused a serious problem in his theory. He also had no necessity to discuss the geometrical meaning of his sum.

3. Dirichlet's theory of definite integral

Now we turn our attention to Dirichlet's 1854 lectures on the definite integral. His lectures began by defining a continuous function as following: $y = f(x)$ is called a continuous single-valued function of x if to every value of x there also corresponds a gradual variation of y i.e. for a fixed x the difference $f(x+h) - f(x)$ convergences to zero for a continuously decreasing h. That is to say, Dirichlet's definition of continuity was almost the same as that of Cauchy, namely not written in term of ε - δ inequalities. Next, Dirichlet demonstrated a fundamental property for continuous functions. This is his statement related to uniform continuity that I have already noted at the beginning of this paper.

This property states that to make the difference of the corresponding value of the function smaller than ρ, one may determine the length of the subinterval σ without noting where the interval exists, i.e. σ is chosen independently of x. In other words, there exists an uniform σ in the interval of the given ρ. In modern terms Dirichlet's fundamental property states that a continuous function on a closed interval becomes uniformly continuous. Thus, the notation of "uniform continuous" was derived through Dirichlet's property, which we prove after defining uniform continuity and compactness today. Then, let's examine his proof of the fundamental property.

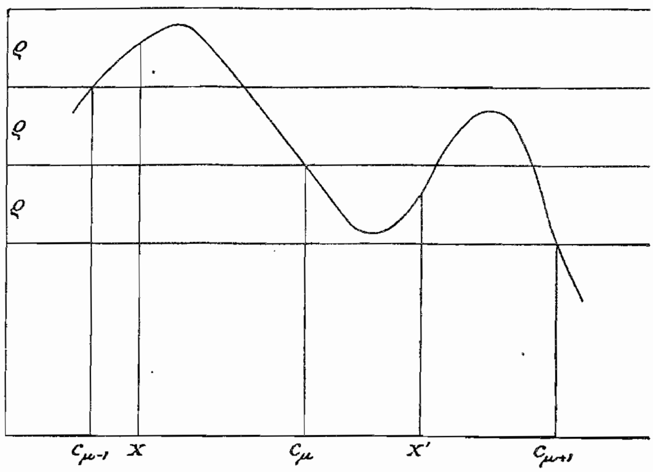
Although, Dirichlet basically accepted Cauchy's idea of the definite integral, he took quite a different approach to it. He used graphs of continuous functions and insisted

[3] If Cauchy had introduced ε- δ inequalities to his discussion, he should have found ε for a given size of interval δ became larger when x approached to the endpoint; δ should have been chosen depending not only on ε but also on x. On the other hand he should have attained an equation such as

$$|S' - S| \leq (x_1 - x_0)\varepsilon_0 + (x_1 - x_0)\varepsilon_1 + \cdots + (X - x_{n-1})\varepsilon_{n-1} \leq \varepsilon'(X - x_0)$$

where ε' is the maximum value of ε_k, instead of equation (4). Since Cauchy recognized ε_k's independence on x, he presumably would have doubted the existence the maximum value.

on geometrical images of his statements. For the fundamental property, he carefully demonstrated it by showing a couple of graphs of continuous functions. In the interval $a \leq x \leq b$ he chose such numbers as $c_0 = a, c_{\mu+1} \geq c_\mu$, that $|f(c_{\mu+1}) - f(c_\mu)| = \rho$ but $|f(x) - f(c_\mu)| \leq \rho$ for all x, $c_\mu < x < c_{\mu+1}$. Then, he indirectly showed that one can choose many but finite numbers of c_μ such that many finitely corresponding subinterval $[c_\mu, c_{\mu+1}]$ covered the interval, and specified that the length of the shortest interval corresponds to σ. His description seems an explanation for the graphs rather than a proof of the property because this property seems trivial as he mentioned. But, he believed that a more detailed explanation should be provided, since one cannot obtain any certain notion of the definite integral without a clear understanding of this property on which the definite integral is constructed. He showed an example of a continuous function, $y = \sin(x^2)$, which is defined from $-\infty$ to ∞. He actually mentioned that it is impossible to define a fixed σ independent of x for a given small ρ, because σ should become smaller with increasing x. Using this fundamental property, he developed his lectures.



Now we try to infer why Dirichlet attained such a sensitive property from his introduction of the definite integral on noting his adoption of ε - δ inequalities in his description. He took a function $f(x)$ to be continuous for x in a given interval with endpoints a and b. Like Cauchy's approach, he divided the interval into n, not necessarily equal parts of $x_1 - a, x_2 - x_1, \cdots, b - x_{n-1}$, multiplied each of them by the value of $f(x)$ at the left-hand point, and thus to form the sum

$$S = \sum_{\mu=0}^{n-1} f(x_\mu)(x_{\mu+1} - x_\mu) \tag{7}$$

where, $a = x_0$ and $b = x_n$. Then, he proved that S has a limit value, which is independent of the way of dividing the interval. Formula (7) is naturally regarded as a rectilinear approximation to curvilinear areas. Differing from Cauchy, Dirichlet explicitly demonstrated this fact by representing the limit concept in terms of $\varepsilon - \delta$ inequalities, because it has been a common procedure to evaluate S using inequalities. His consistent usage of ε - δ inequalities was an appropriate choice for his demonstration of the geometrical meaning of a definite in·egral.

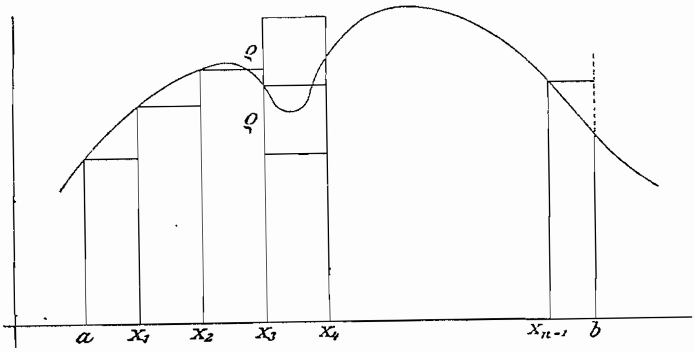
In his proof Dirichlet assumed that for a given ρ, one can choose such σ that makes $|f(x_{\mu+1})-f(x_\mu)| < \rho$ if, the length of every subintervals $[x_\mu, x_{\mu+1}]$ becomes shorter than σ. In other words for any small ρ, we can chose an appropriate σ independent to x. Because of this fundamental property, every continuous function on a closed interval satisfies his assumption. Because for any μ the difference between $f(x_\mu)$ and $f(x_{\mu+1})$ on the interval is smaller that ρ, he derived the relation

$$(x_{\mu+1} - x_\mu)(f(x_\mu) - \rho)) \quad < A_\mu \quad < (x_{\mu+1} - x_\mu)(f(x_\mu) + \rho)) \tag{8}$$

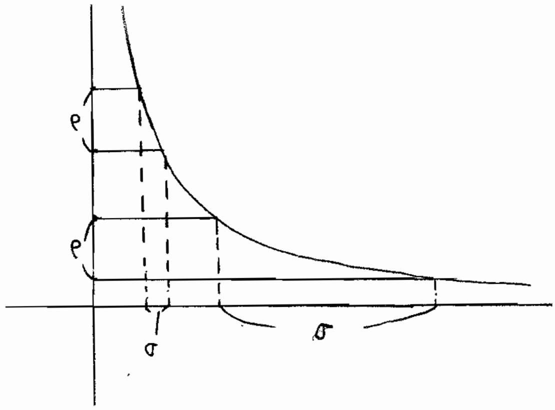
where, A_μ is the area surrounded with curves of $x = x_\mu, x = x_{\mu+1}, y = 0, y = f(x)$, by using a graph of a continuous function. Taking the sum from $\mu = 0$ to n, he obtained

$$S - \rho(b-a) < A < S + \rho(b-a), \quad \text{or} \quad A - \rho(b-a) < S < A + \rho(b-a) \tag{9}$$

where, A is the area surrounded by these curves with $x = a$, $x = b$, $y = 0$, $y = f(x)$. This fact implys that if one makes each interval shorter and shorter, then the difference $\rho(b-a)$ approaches 0, i.e. S indefinitely approaches A. By using his fundamental property he succeeded in demonstrating the geometrical meaning of Cauchy's sum.



Although, Dirichlet didn't explicitly mention it, a following mathematical fact, which is easily found through his graph for a continuous function, which presumably had brought him to the idea of his fundamental property. As an example of the case where σ must be chosen dependent of x for a given ρ, let's consider $y = \dfrac{1}{x}$ on $(0, b)$, where b is any positive value of x. For this function, σ should be chosen smaller and smaller with decreasing x for a given ρ. Thus, we need a infinite number of subintervals to cover the interval. In this case the definite integral cannot be defined because some finite division of $[a, b]$ should be determined at the beginning even if the number of subdivision finally becomes infinite. That is to say, his definition of definite integrals becomes valid when the continuous function is uniformly continuous. While preparing for these lectures, Dirichlet presumably had described many graphs of several kinds of continuous functions, had tried to decide σ for each cases, and had realized the requirement of uniform continuity. Fortunately, any continuous function on a closed interval is uniformly continuous. He actually realized the importance of this fact. Then he devoted considerable time to explain the fundamental property. Through this process Dirichlet succeeded in attaining a relatively more precise understanding of the property of a continuous function than Cauchy, while Dirichlet accepted almost the same definitions of continuity and a definite integral as given by Cauchy.



4. Discussion

Our examination indicates that it is the essential difference between Cauchy and Dirichlet where Dirichlet introduced his integral theory by demonstrating the geometrical meaning of Cauchy's sum, which Cauchy had not done. To do this, Dirichlet described graphs and decided to consistently use epsilontic representations. Because of the adoption of epsilontics he found that the condition of uniform continuity was required for Cauchy's definition of a definite integral. This situation made Dirichelt decide to demonstrate his fundamental property that involved the notion of uniform continuity. Therefore, it is essential for our questiȯn how Dirichlet decided to confirm the geometrical meaning of Cauchy's sum.

Dirichlet's example of a function $y = \sin(x^2)$ suggests to us that he noticed that this function made him decide to prove the meaning of Cauchy's definite integral. The behavior of this oscillating function is quite strange: the more x increases, the more frequently the function changes. If Cauchy had obtained such a kind of function, he should have checked if Cauchy's sum could actually be constructed. But, Cauchy's example didn't involve such a curious oscillating function although he integrated many functions. From a comparison with two mathematicians' examples, we may conclude that Dirichlet's encounter with this strange oscillating function seemed to be an essential factor for his derivation to the fundamental property.

Now we turn our attention to how Dirichlet attained this function and if he had an intention to integrate it. In his paper of 1835 entitled "Sur l'usage des intégral definies dans la sommantion des séries finies ou infinies" where he discussed some problem with the number theory, he derived the following result: "If sum of finite or infinite series of

$$F(\alpha) = c_0 + c_1 \cos\alpha + c_2 \cos 2\alpha + \dots \qquad (10)$$

is known, one can always represent the sum of the new series

$$c_0 + c_1 \cos 1^2 \cdot \frac{2\pi}{n} + c_2 \cos 2^2 \cdot \frac{2\pi}{n} + \dots,$$

$$c_0 + c_1 \sin 1^2 \cdot \frac{2\pi}{n} + c_2 \sin 2^2 \cdot \frac{2\pi}{n} + \dots, \tag{11}$$

using the function $F(\alpha)$. " In proving this theorem he actually set

$$\int_{-\infty}^{\infty} \cos(\alpha^2) d\alpha = a, \qquad \int_{-\infty}^{\infty} \sin(\alpha^2) d\alpha = b. \tag{12}$$

Furthermore, he repeatedly used

$$\int_{-\infty}^{\infty} \cos(\alpha^2) d\alpha = a \tag{13}$$

in his paper of 1837, "Recherches sur diverses applications de l'analyse infinitésimale à la théorie des nombres." Thus, he couldn't help neglecting this function while demonstrating his general theory for the definite integral.

Dirichlet, who had met curious functions, presumably felt that the geometrical meaning of Cauchy's sum should be demonstrated again. For the purpose of evaluating the area, he rewrote Cauchy's definition for the definite integral and some properties of the continuous function in terms of ε - δ inequalities. In this process he found that uniform continuity was required for defining Cauchy's sum. Thus, Dirichlet demonstrated that a continuous function on a closed interval becomes uniformly continuous, namely his fundamental property. It is his encounter with the special function that consequently brought him the idea of this property.

As I mention at the beginning, there should be the other factors that gave him an idea of the fundamental property. It is our next task to find them and to understand the importance of his encounter with the curious function in the process of obtaining this property.

It is worth noting in passing that in Dirichlet's discussion, the fundamental property was rather more essential than the notion of uniformly continuity itself. But, by noting the fundamental property and "a strange oscillating function," mathematicians seemed to arrive at Heine's establishment of the notion of uniform continuity in 1872. In any case, it is certain that Dirichlet definitely played an essential role in establishing this notion.

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