確率超過程と超汎関数

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1 Introduction

Some aspects of generalized function theory started to appear in mathematics in the nineteenth century. We can see it in the definition of the Green's function, in the Laplace transform, and in Riemann's theory of trigonometric series, which were not necessarily the Fourier series of an integrable function.

The Laplace transform is intensively used in engineering and it leads to use symbolic methods which are called later operational calculus. There are used divergent series, these methods are not accepted from the point of view of pure mathematics. Later they are typical application of generalized function methods.

After the Lebesgue integral was introduced, a concept of generalized function became essential to mathematics. An integrable function, in Lebesgue's theory, is equivalent to any other which is the same almost everywhere. That means its value at a given point is not its most important feature. An evident formulation is given, in functional analysis, of the essential feature of an integrable function, such as the way it defines a linear functional on other functions. This allows a definition of weak derivative.

The Dirac delta function was defined by Paul Dirac; this was to treat measures, thought of as densities. Sobolev who was working in partial differential equation theory, defined the first suitable theory of generalized functions, from the mathematical view point, in order to work with weak solutions of partial differential equations.

Schwartz distributions

The theory of distributions was developed by Laurent Schwartz. It is based on duality theory for topological vector spaces.

The theory of distribution is widely affect the differential and integral calculus. Heaviside and Dirac had generalized the calculus with specific applications in mind, and other similar methods of formal calculation, however, profound mathematical foundation was not given. Schwartz developed the theory of distributions by putting methods of this type onto a thourough basis. The theory extended their range of application, providing powerful tools for applications in numerous areas.

2 Laurent Schwartz (1915-2002)

Laurent Schwartz entered the Ecole Normale Superieure in Paris in 1934 and graduated with the Agreation de Mathematique in 1937 and studied for his doctorate in the Faculty of Science at Strasbourg which he was awarded in 1943.

His teachers were Choquet, Frechet, Borel, Julia, Cartan, Lebesgue, Hadamard. Schwartz writes, "The life of the ENS was a marvel for a young person of my temperament. In one blow, the field of mathematics became infinitely wide."

Schwartz was lecturer at the Faculty of Science at Grenoble the year 1944-45 before moving to Nancy where he became a professor at the Faculty of Science. During this period he produced his famous work on the theory of distributions.

From 1953 to 1959, Schwartz was holding the position of Professor in Paris. He taught at the Ecole Polytechnique in Paris from 1959 to 1980. He then spent three years at the University of Paris VII before he retired in 1983.

Schwartz made the outstanding contribution in the theory of distributions to mathematics in the late 1940s. He published these ideas in the paper "*Generalisation de la notion de fonction, de derivation, de transformation de Fourier et applications mathematique et physiques* " in 1948.

And the other literatures on Theory of distributions are

Théoriedes distributions, Tome I. 1950, Tome II. 1951, Hermann & Cie Paris.

Théoriedes distributions, Nouvelle ed. 1966 Hermann

Schwartz got a Fields Medal, presented by Harald Bohr, at the International Congress in Harvard on 30 August 1950 for his work on the theory of distributions. Schwartz has received a long list of prizes, medals and honours in addition to the Fields Medal.

His autobiography "Un mathématicien aux prises avec le siècle. Editions odile Jacob. 1997, (Japanese translation by 弥永健一、闘いの世紀を生きた数学者・上、下、シュプリンガー・ジャパン、2006.) tells us how he had been living as a mathematician. We are surprised to see what we have never imagined.

Literatures below are specifically mentioned.

Geometry and probability in Banach space. LNM 852, Springer, 1981

We are particularly interested in his work on probability theory. Also see, Notice of AMS 50, no.9 (2003), 1072-1084.

The contents and his basic idea are found in those literature. We shall however mention some of his results, which are often used in white noise theory, in the next section.

3 Generalized function (Distribution)

The basic idea of generalized function is as follows. If $f : R \to R$ is an integrable function, and $\phi : R \to R$ is a smooth function with compact support, then $\int f\phi dx$ is a real number which depends on ϕ linearly and continuously. The function f can be thought as a continuous linear functional on the test functions space of ϕ.

This notion of **continuous linear functional on the space of test functions** is therefore used as the definition of a distribution.

Such distributions may be multiplied with real numbers and can by added together, so they form a real vector space. In general it is not possible to define a multiplication for distributions, but distributions may be multiplied with infinitely often differentiable functions.

To define the derivative of a distribution, we first consider the case of a differentiable and integrable function $f : R \to R$. If ϕ is a test function, then we have

$$\int_R f'\phi\, dx = -\int_R f\phi'\, dx$$

using integration by parts (note that ϕ is zero outside of a bounded set and that therefore no boundary values have to be taken into account). This suggests that if S is a distribution, we should define its derivative S' as the linear functional which sends the test function ϕ to -$S(\phi')$. It turns out that this is the proper definition; it extends the ordinary definition of derivative, every distribution becomes infinitely often differentiable and the usual properties of derivatives hold.

The Dirac delta function is the distribution which sends the test function ϕ to $-\phi'(0)$. It is the derivative of the

step function

$$H(x) = \begin{cases} 0, & \text{if } x < 0, \\ 1, & \text{if } x > 0 \end{cases}$$

The derivative of the Dirac delta is the distribution which sends the test function ϕ to -$\phi'(0)$.

Compact support and convolution

We say that a distribution S has compact support if there is a compact subset $K$of U such that for every test function ϕ whose support is completely outside of K, we have $S(\phi) = 0$. Alternatively, one may define distributions with compact support as continuous linear functionals on the space $C^\infty(U)$; the topology on $C^\infty(U)$ is defined such that ϕ_k converges to 0 if and only if all derivatives of ϕ_k converge uniformly to 0 on every compact subset of U.

If both S and T are distributions on R^n and one of them has compact support, then one can define a new distribution, i.e. the convolution $S*T$ of S and T. It generalizes the classical notion of convolution of functions and is compatible with differentiation in the following sense:

$$\frac{d}{dx}(S*T) = (\frac{d}{dx}S)*T + S*(\frac{d}{dx}T).$$

Tempered distributions and Fourier transform

By using a larger space of test functions, one can define the tempered distributions, a subspace of $D'(R^n)$. These distributions are useful if one studies the Fourier transform in generality: all tempered distributions have a Fourier transform, but not all distributions have one.

The space of test functions employed here, the so-called Schwartz-space, is the space of all infinitely differentiable rapidly decreasing functions, where $\phi : R^n \rightarrow R$ is called rapidly decreasing if any derivative of ϕ, multiplied with any power of $|x|$, converges towards 0 for $|x| \rightarrow \infty$. These functions form a complete topological vector space with a suitably defined family of seminorms. More precisely, let

$$p_{\alpha,\beta}(\phi) = \sup_{x \in \mathbf{R}^n} |x^\alpha D^\beta \phi(x)|$$

for α, β multi-indices of size n. Then ϕ is rapidly-decreasing if all the values

$$p_{\alpha,\beta}(\phi) < \infty$$

The family of seminorms $p_{\alpha,\beta}$ defines a locally convex topology on the Schwartz-space. It is metrizable and complete.

The derivative of a tempered distribution is again a tempered distribution. Tempered distributions generalize the bounded locally integrable functions; all distributions with compact support and all square-integrable functions can be viewed as tempered distributions.

To study the **Fourier transform**, it is best to consider complex-valued test functions and complex-linear distributions. The ordinary continuous Fourier transform F yields then an automorphism of Schwartz-space, and we can define the Fourier transform of the tempered distribution S by $(FS)(\phi) = S(F\phi)$ for every test function, ϕ, FS is thus again a tempered distribution. The Fourier transform is a continuous, linear, bijective operator from the space of tempered distributions to itself. This operation is compatible with differentiation in the sense that

$$F(\frac{d}{dx}S) = ixFS$$

and also with convolution: if S is a tempered distribution and ψ is a slowly increasing infinitely often differentiable function on R^n (meaning that all derivatives of ψ grow at most as fast as polynomials), then $S\psi$ is again a tempered distribution and

$$F(S\psi) = FS * F\psi.$$

4 Rigged Hilbert space (Gel'fand triple)

The rigged Hilbert space appears in white noise analysis in various places, where the expression has variation depending on the purpose.

In mathematics, a rigged Hilbert space (Gel'fand triple, nested Hilbert space, equipped Hilbert space) is a construction designed to link the distribution and the test function, where the square-integrable aspects of functional analysis serves as a key role.

Since a function such as

$$x \mapsto e^{ix},$$

which is in an obvious sense an eigenvector of the differential operator

$$i\frac{d}{dx}$$

on the real line R, is not square-integrable for the usual Borel measure on R, this requires some way of stepping outside the strict confines of the Hilbert space theory. This was supplied by the apparatus of Schwartz distributions, and a generalized eigenfunction theory was developed in the years after 1950.

Functional analysis approach

The concept of rigged Hilbert space places this idea in abstract functional-analytic framework. Formally, a rigged Hilbert space consists of a Hilbert space H, together with a subspace Φ which carries a finer topology, that is one for which the natural inclusion

$$\Phi \subset H$$

is continuous. It is no loss to assume that Φ is dense in H for the Hilbert norm. We consider the inclusion of dual spaces H^* in Φ^*. The latter, dual to Φ in its 'test function' topology, is realised as a space of distributions or generalised functions of some sort, and the linear functionals on the subspace Φ of type

$$\phi \mapsto \langle v, \phi \rangle$$

for v in H are faithfully represented as distributions (because we assume Φ dense).

Now by applying the Riesz representation theorem we can identify H^* with H. Therefore the definition of rigged Hilbert space is :

$$\Phi \subset H \subset \Phi^*.$$

The most significant examples are for which Φ is a nuclear space; this comment is an abstract expression of the idea that Φ consists of test functions.

Formal definition

A rigged Hilbert space is a pair (H, Φ) with H a Hilbert space, Φ a dense subspace, such that Φ is given a topological vector space structure for which the inclusion map i is continuous. Identifying H with its dual space H^*, the adjoint to i is the map $i^* : H = H^* \mapsto \Phi$. The duality pairing between Φ and Φ^* has to be compatible with the inner product on H :

$$\langle u, v \rangle_{\Phi \times \Phi^*}.$$

whenever $u \in \Phi \subset H$ and $v \in H = H^* \subset \Phi^*$.

$$\Phi \subset H = H^* \subset \Phi^*.$$

Note that even though Φ is isomorphic to Φ^* if Φ is a Hilbert space in its own right, this isomorphism is not the same as the inclusion .

Example. Fourier integral

$$f(x) = \int_R e^{isx} \hat{f}(s) ds, \quad x \in R, \ f, \ \hat{f} \in L^2(R).$$

The system $\{e^{isx}, s \in R\}$ is a system of generalized eigenfunctions of the differentiation operator, acting on $L^2(R)$, arising under the natural rigging of this space by the Schwartz space $S(R)$.

These Hilbert spaces play an important role in the definition of generalized white noise functinal.

5 Hyperfunction

Hyperfunctions are sums of boundary values of holomorphic functions, and can be thought of as distriburtions of infinite order. Hyperfunctions were introduced by Miko Sato in 1958, building upon earlier work by Grothiendieck and others.

The distribution theory led to investigation of the idea of hyperfunction, in which spaces of holomorphic functions are used as test functions. A refined theory has been developed, in particular by Mikio Sato, using sheaf theory and several complex variables. This extends the range of symbolic methods that can be made into rigorous mathematics, for example Feynman integrals.

A hyperfunction is specified by a pair (f, g), where f is a holomorphic function on the lower half-plane and g is a holomorphic function on the upper half-plane. Informally, the hyperfunction (f, g) is the sum of the boundary values of f and g. If f is holomorphic on the whole complex plane, then it should have the same boundary values when considered as a function on either the upper or lower hyperplane.

For further reference, see

佐藤幹夫、超関数の理論、数学 10 巻、1-27.

6 Generalized stochastic process

White noise analysis started in 1975, more than three decades ago; this means it is now in the history. However, there are many serious misunderstandings. Anyhow it is a good opportunity to have a review of the history of white noise analysis. Before white noise, there is a history of generalized stochastic process.

Well known approaches

1) Stationary random distributions : by K. Itô. Mem. Univ. of Kyoto. Math. 1953, 209-223.

The stationary distributions were introduced as a generalization of stationary stochastic process and they are classified according to the specral density.

2) Generalized random processes : by Gel'fand. He defined a stochastic process, a sample function of which is a generalized function. Doklady Acad Nauk, 1955. 853-856.

   Literature : Gel'fand and Shirov, Generalized function Vol 4, 1955. Academic Press.

3) Generalized white noise function (started in 1975)

There are two ways to introduce a space of generalized white noise functionals. One is using the method of the Gel'fand's generalized functions (Hida 1975). The other is an infinite dimensional analogue of Schwartz distributions (1980- , Kubo-Takenaka then Potthoff-Streit).

The first method uses the Sobolev space structure which is familier for us, while the second method provides a characterization theorem of generalized white noise functionals in such a way that, let F be its S-transform, then it is necessary and sufficient that it is ray entire and it satisfies

$$|F(z\xi)| \le K_1 \exp[K_2|z|^{\,2}|\xi|_p^2].$$

Generalized functionals of white noise (Hida distribution)

White noise analysis started in 1975, more than three decades ago, where the time derivative $\dot{B}(t)$ of a Brownian motion $B(t)$ was introduced as a variable of white noise functionalsby T. Hida. Since $\dot{B}(t,\omega)$ is no more an ordinary random variable but an *idealized generalized random variable*. However, by many reasons $\dot{B}(t)$ is taken to be an elemental (atomic) variable, so that a sharp time description is given. He does not smear the $\dot{B}(t)$ by a smooth function ξ like that $\dot{B}(\xi) = -\int \xi'(t)B(t)dt$. He gives a rigorous meaning to each $\dot{B}(t)$ by introducing a new space of generalized random variable. It is therefore noted that the $\dot{B}(t)$ can not be approximated by members in the usual space of Brownian functionals.

For a general setup of the space of white noise functionals, $\dot{B}(t)$'s are taken to be variables, so that their functionals can be defined, not formal but in a correct manner. By doing so, we can see many applications, for instance, in the expression of propagator in quantum dynamics according to the Feynman integral.

To define $\dot{B}(t)$ for every t rigorously, He use the rigged Hilbert space in the following manner:

Start with smeared variables

$$\langle \dot{B}, \xi \rangle = \int \xi(t)\dot{B}(t)dt,$$

ξ being a member of E, say the Schwartz space.d Then extend ξ to be in the Hilbert space $L^2(R)$ to have

$$H_1 = \left\{\dot{B}(f), f \in L^2(R)\right\} \cong L^2(R) \qquad (1)$$

since $\dot{B}(f) = \langle \dot{B}, f \rangle$ is an ordinary Gaussian random variables $N(0, |f|^2)$.

Then, take a rigged Hilbert space

$$K \subset L^2(R) \subset K^*,$$

where K is taken so as K^* involves delta-functions, and the isomorphism (1) extends to a rigged Hilbert space

$$H_1^{(1)} \subset H_1 \subset H_1^{(-1)},$$

where $\dot{B}(t)$ is a well defined member of $H_1^{(-1)}$.

Note. It should be made clear that a delta function is not a member of $L^2(R)$, but it belongs to much wider class K^*. Similarly $\dot{B}(t)$ has an identity as a member of $H_1^{(-1)}$ and can not be reduced to an element of H_1.

Further, we can form functionals $\varphi(\dot{B}(t), t \in R)$ to use a rigged Hilbert space

$$H_n^{(n)} \subset H_n \subset H_n^{(-n)},$$

where H_n is a space of homogeneous chaos and $H_n^{(-n)}$ involves Hermite polynomials in $\dot{B}(t)$'s of degree n, which are rigorously defined.

The space of generalized white noise functionals is definded by

$$(L^2)^- = \bigoplus_{n\geq 0} H_n^{(-n)}.$$

As for the second method of defining the space of generalized white noise functionals (Hida distributions) denoted by $(S)^*$ can be defined as a member of the rigged Hilbert spaces

$$(S) \subset (L^2) \subset (S)^*,$$

where (L^2) is the space of ordinary white noise functionals and

$$(L^2) = \oplus H_n$$

is a Fock space. Actual method to form (S) uses the second quantization method, where the operator is $A = -\frac{d^2}{du^2} + u^2 + 1$.

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[5] T. Hida, Analysis of Brownian functionals. Carleton Math. Notes 1975. Carleton Univ.

[6] T. Hida et al, White noise. An infinite dimensional calculus. Kluwer Academic Pub. 1993.