THE ORIGINAL NAVIER-STOKES EQUATIONS AND EQILIBRIUM EQUATIONS OF FLUID.
原型の NAVIER-STOKES 方程式と流体平衡方程式

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ABSTRACT. After Laplace's death in 1827, to Laplace, Gauss[10, 11] critisize severely the errors of Laplace[18]. Before and after that, Poisson[31], Green[12] and Stokes[37] discover each formulae of the triple integral. These are deduced from studying the equation of the fluid equiliblium, or the meniscus, or the rays. Gauss solve menuscus and Hamilton the systems of rays respectively, with the least actions or the calculas of variations. Hamilton[13] presents the partial differential equations using tensor, which is the same as Helmholtz's one in his vorticity equations.

On the other hand, it was at the same time when the Navier-Stokes equations [*NS* equations] are formulated by Navier[26](1827), Cauchy[5](1828), Poisson[29](1831), Saint-Venant[36](1843), Stokes[37](1849), and stated the linear fluid-dynamic equations, which are now so-called Stokes equations without not only viscosity but the name of equation, by Maxwell[24](1865), Rayleigh[34](1883), Raynolds[35](1883), Boltzmann[3](1895), Prandtl[32](1905).

At latest, the fluid-dynamic equations composed of the both linear term and nonlinear viscosity-term ware cited by Prandtl[33](1934) under the name of the *NS* equations.

We would like to discuss the topics relating to the original, microscopically-descriptive [*MD*] *NS* equations and fluid mechanics in Gauss' Latin papers as a contemporary with the promugators of the *NS* equations. We show the essence of common handling of *MD* equations among them.

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Key words: the Navier-Stokes equations, the eqilibrium equations of fluid, hydrostatics, the microscopica descriptive equation, the "two-constant theory", meniscus, capillarity, capillary action, curvature, the variation problem, Gauss' formula of integration.

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[1] Laplace(1749-1827), Gauss(1777-1855), Navier(1785-1836), Cauchy(1789-1857), Poisson(1781-1840), Saint-Venant(1797-1886), Stokes(1819-1903).

1. Preliminary

Gauss didn't mention the following fact, and Bowditch [2] also didn't comment on Gauss's work in Laplace's total works[18] except for only one comment of the name "Gauss" [18, p.686]. [3]

N.Bowditch comments as follows :

> This theory of capillary attraction was first published by La Place in 1806 ; and in 1807 he gave a supplement. In neither of these works is the repulsive force of the heat of fluid taken into consideration, because he supposed it to be unnecessary. But in 1819 he observed, that this action could be taken into account, by supposing the force $\varphi(f)$ to represent the difference between the attractive force of the particles of the fluid $A(f)$, and the repulsive force of the heat $R(f)$ so that the combined action would be expressed by, $\varphi(f) = A(f) - R(f)$; $\cdots$ [18, p.685].

We would like to pay attention to Bowditch's remark about the works of Gauss and Poisson as follows :

> In 1830, Gauss published a work on capillary attraction entitled "*Principia generalia theoriœ figurœ fluidorum in statu equilibrii, etc.,*" ("*General principle of theory of the figure of fluid in state equilibrium*"), where, by means of the principle of virtual velocities, he obtains the figure of the capillary surface, and other theorems as they are given by La Place in this volume, and he also gives a more complete demonstration of the constancy of the angle of contact of the fluid with the sides of the tube. Finally, M.Poisson, in 1831, published his "*Nouvelle Théorie de l'action capillaire, etc.,*"("*New theory of the capillary action*"), where he expressly introduces into the formulas the consideration of the change *of density of the fluid at its surface and near the sides of the tube* in consequence of the corpusclar attraction. [18, p.686]

In his historical descriptions about the study of capillariy action, we would like to recognize that there is no counterattack to Gauss, but the correct valuation. Gauss [11] stated his conclusion about Laplace's paper as follows :

> At hancce propositionnem cardinalem totius theoriae per calculum demonstrare ne suscepit quidem ill. Laplace ; quae enim in dissertatione priori p.5 huc spectantia afferuntur, argumentationem vagam tantummode exhibent et quad demonstrandum erat iam supponunt : calculi autem p.44 sq. suscepti effectu carent.
>
> (Trans.) To this cardinal proposition of the total theory with calculation for demonstration, we can not accept the papers by Mr. Laplace ; in p.5, since not only he developed clearly incorrect argument but also showed even the false proofs : we consider that his calulations in the pages, p.44 and the followings it, [4] have non effect in vain. [11, p.33-34]

2. Laplace and Gauss

2.1. Laplace's theory of the capillary action.

2.1.1. Laplace's conclusions of theory of the capillary action.

Laplace stated his "*complete theory*" of capillary action in the introduction of [17], which consisted of the first part and the supplemental part. Among them, we show some paragraphs, which he stated his motivations, in the introduction of the first part as follows :

> J'ai cherché, il y a longtemps, à determiner les lois d'attraction qui représentent ces phénomènes : de nouvelles recherches m'ont enfin conduit à faire voir qu'ils sont tous représentés par les mêmes lois qui satisfont aux phénomès de la réfraction, c'est-à-dire par les lois dans lequelles l'attraction n'est sensible qu'à des distances insensibles; et il en résulte une théorie complète de l'action capillaire.[17, p.2]
>
> . . .

[2] The present work is a reprint, in four volumes, of Nathaniel Bowditch's English translation of volumes I, II, III and IV of the French-lamguage treatise *Traité de Mécanique Céleste* by P.S.Laplace. The translation was originally published in Boston in 1829, 1832, 1834, and 1839, under the French title, "*Mécanique Céleste*", which has now been changed to its English-language form, "*Celestial Mechanics.*"

[3] Wowditch's comment number [9173g].

[4] There are 35 pages of calculation between p.44 and p.78 in his *Supplement*.

De ces résultats relatifs aux terminés par des segmens sensibles des surface sphérique, je conclus ce théorème général : ≪ Dans toutes les loi qui rendent l'attraction insensible á des distances sensibles, l'action d'un corps terminé par une surface courbe, sur un canal intérieur infiniment étroit, perpendicularire à cette surface dans un point quelconque, est égale à la demi-somme des actions sur le même canal, de deux sphères qui auraient pour rayons le plus grand et le plus petit des rayons osculateurs de la surface, à ce point ≫. [17, p.4]

From the translation by Bowditch[18], for brevity, we show the corresponding part with above as follows :

A long while ago, I endevored in vain to determine the laws of attraction which would represent these phenomena ; but same late researches have rendered it evident that the whole may be represented by the same laws, which satisfy the phenomena of refraction ; that is, by laws in which the attraction is sensible only at insensible distances ; and from this principle we can deduce a complete theory of capillary attraction. [18, p.688]

...

From these results, relative to bodies terminated by sensible segments of a spherical surface, I have deduced this general theorem. *"In all the laws which render the attraction insensible at sensible distance, the action of body terminated by a curve surface, upon an infinitely narrow interior canal, which is perpendicular to that surface, at any point whatever, is equal to the half sum of the actions upon the sema canal, of two spheres which have the same radii as the greatest and the least radii of curvature of the surface at that point."* By means of this theorem, and of the laws of the equilibrium of fluids, we can determine the figure which a fluid must have, when it is included whithin a vessel of a given figure, and acted upon by gravity. [18, p.689]

2.1.2. Laplace's theory of the capillary action.

Laplace's theories of the capillary action are described in the 14 articles. We cite only the contents of *no* 1 of theory of [17] pointed out by Gauss:

¶ *no* 1 of the theory of capillary action :

Considérons vase $ABCD$ (*fig. 1*), [5] plein d'eau jusqu'en AB, et concevons un tube capillaire de verre, $NMEF$, extrémité inférieure; l'eau s'élevera dans le tube jusqu'en O, et sa surface prendra la figure concave NON, O étant le point le plus bas de cette surface. Imaginons par ce point et par l'axe du tube, un filet d'eau renfermé dans un canal infiniment étroit $OZRV$; il est clair, d'après le principe que nous venons d'exposer sur le peu d'étendue des attractions capillaires, que l'action de l'eau inférieure à l'horizontale IOK, sera la même sur la colonne OZ, que l'action du vase la colonne VR. Mais le ménisque $MIOKN$ agira sur la colonne OZ de bas en haut, et tendra parconséquent à soulever le fluide. Ainsi, dans l'état d'équilibre, l'equ du canal $OZRV$ devra être plus élevée dans le vase, pour compenser par son pois, cette action du ménisque.

Soit r la distance du point attiré, au centre d'une couche sphérique dont u est le rayon et du l'épaisseur. Soir encore θ l'angle que le rayon u fait avec la droit r, ϖ l'angle que la plan qui passe par les deux droites r et u fait avec un plan fixe passant par la droite r : l'élément de la couche sphérique sera $u^2du.d\varpi.d\theta.\sin.\theta$. Si l'on nomme ensuite f la distance de ce élément, au point attiré que nous supposerons extérieur à la couche; nous aurons

$$f^2 = r^2 - 2ru.\cos.\theta + u^2.$$

Représentons par $\varphi(f)$ la loi de l'attraction à la distance f, attraction qui, dans le cas présent, est insensible lorsque f a une valeur sensible; l'action de l'élément de la couche sur le point attiré, décomposée parallélement à r, et dirigée vers le centre de la couche, sera

$$u^2du.d\varpi.d\theta.\sin.\theta.\frac{r-u.\cos.\theta}{f}.\varphi(f)$$

[5] The original *fig. 1* by Laplace [17] is shown in the last page of our paper.

$$\frac{r - u.\cos.\theta}{f} = \frac{df}{dr}$$

$$u^2 du.d\varpi.d\theta.\sin.\theta.\frac{df}{dr}.\varphi(f)$$

Désignons par $c - \Pi(f)$, l'intégrale $fdf.\varphi(f)$, prise depuis $f = 0$; c étant la valeur de cette intégrale, lorsque f est infini; $\Pi(f)$ sera une quantité positive décroissante avec une extrême rapidité; de manière à devenir insensible, lorsque f a une valeur sensible. [17, p.11]

¶ *no* 4 of the theory of capillary action :

Soit O(*fig. 3*) [6] la plus bas de la surface AOB de l'eau renfermée dans un tube. Nommonz z la coordinnée verticale OM; x et y, les deux coordinées horizontales d'un point quelconque N de la surface. Soient R et R' le plus grand et le plus petie des rayons osculateurs de la surface à ce point.

R et R' seront les deux racines de l'équation

$$R^2.(rt - s^2) - R.\sqrt{(1 + p^2 + q^2)}.\{(1 + q^2).r - 2pqs + (1 + p^2).t\} + (1 + p^2 + q^2)^2 = 0, \tag{1}$$

équation dans laquelle

$$p = \frac{dz}{dx}; \quad q = \frac{dz}{dy}; \quad r = \frac{d^2 z}{dx^2}; \quad s = \frac{d^2 z}{dxdy} = \frac{dp}{dy} = \frac{dq}{dx}; \quad t = \frac{d^2 z}{dy^2}. \tag{2}$$

On aura donc

$$\frac{1}{R} + \frac{1}{R'} = \frac{(1 + q^2).\frac{dp}{dx} - pq.\left(\frac{dp}{dy} + \frac{dq}{dx}\right) + (1 + p^2).\frac{dq}{dy}}{(1 + p^2 + q^2)^{\frac{3}{2}}} = \frac{(1 + q^2).r - 2pqs + (1 + p^2).t}{(1 + p^2 + q^2)^{\frac{3}{2}}} \tag{3}$$

$$K - \frac{H}{2}.\left(\frac{1}{R} + \frac{1}{R'}\right) + gz = K - \frac{H}{2}.\left(\frac{1}{b} + \frac{1}{b'}\right); \quad \Rightarrow \quad \left(\frac{1}{R} + \frac{1}{R'}\right) - \frac{2gz}{H} = \frac{1}{b} + \frac{1}{b'}; \tag{4}$$

b et b' étant le plus grand et le plus petit des rayons osculateurs de la surface au point O, et g étant la pesanteur. En effet, l'action du fluide sur le canal, au point N, est par ce qui précéde, $K - \frac{H}{2}.\left(\frac{1}{R} + \frac{1}{R'}\right)$; et de plus, la hauteur du point N audessus du point O est z. L'équation précédente donne, en y substituant pour $\frac{1}{R} + \frac{1}{R'}$, sa valeur, [7]

$$(a) \qquad \frac{(1 + q^2).r - 2pqs + (1 + p^2).t}{(1 + p^2 + q^2)^{\frac{3}{2}}} - \frac{2gz}{H} = \frac{1}{b} + \frac{1}{b'}; \tag{5}$$

2.1.3. Laplace's supplement for theory of the capillary action.

Laplace stated the supplemet under the title of *Nouvelle manière de considérer l'action capillaire* in [17]. We show the original contents of 5 and 18 page of [17] pointed out by Gauss. These translations are in Bowditch[18] [8], however, omitted for lack of space.

¶ 5 page of *Supplement* :

(1) L'intégrale relative à f peut être prise depuis $f = 0$ jusqu'à f infini; ensorte qu'elle est indépendente des dimensions de la masse attirante. C'est là ce qui caractérise ce genre d'attractions qui n'étant sensibles qu'à des distances imperceptibles, permettent d'ajouter ou de négliger à volonté, les attractions des corps, à des distances plus grandes que le rayon de leur sphère d'activité sensible.

(2) Désignons comme dans le n^o 1 de ma Théorie de l'action capillaire, par $c - \Pi(f)$, l'intégrale $fdf.\varphi(f)$, prise depuis $f = 0$; c étant la valeur de cette intégrale, lorsque f est infini. $\Pi(f)$ sera une quantité positive décroissante avec une extrême rapidité; et l'on aura, en prenant les intégrales depuis $f = 0$,

$$\int f^4 df.\varphi(f) = -f^4.\Pi(f) + 4\int f^3 df.\Pi(f).$$

[6] The original *fig. 3* by Laplace [17] is shown in the last page of our paper.

[7] From (3) and (4) we get it.

[8] In this translation by Bowditch[18], the relation with the original pages are not showed.

$-f^4.\Pi(f)$ est nul, lorsque f est infini; car, quoique f^4 devienne alors infini, l'extrême rapidité avec laquelle $\Pi(f)$ est supposé décroître, rend $f^4.\Pi(f)$ nul.

(3) Les functions $\varphi(f)$ et $\Pi(f)$ ne peuvent être mieux comparées qu'à des exponentielles telles que c^{-if}, c étant le nombre dont le logarithme hyperbolique est l'unité, et i étant un très-grand nombre.

(4) En effet, c^{-if} est fini lorsque f est nul, et devient nul lorsque f est infini; de plus, il décroît avec une extrême rapidité, et le produit $f^n.c^{-if}$ est toujours nul, quel que soit l'exposant n, lorsque f est infini.

(5) Soit encore, comme dans le n^o 1 de la Théorie citée,

$$\int fdf.\Pi(f) = c' - \Psi(f); \tag{6}$$

c' étant la valeur de cette intégrale, lorsque f est infini. $\Psi(f)$ sera encore une quantité positive décroissante avec une extrême rapidité; et l'on aura

$$4.\int f^3df.\Pi(f) = -4f^2.\Psi(f) + 8.\int fdf.\Psi(f).$$

dans le cas de f infini, $f^2\Psi(f)$ devient nul; on a donc en prenant l'intégrale depuis $f = 0$, jusqu'à f infini,

$$4\int f^3df.\Pi(f) = 8\int fdf.\Psi(f).$$

(6) Einfin, si l'on désigne, comme dans le *no* cité, par $\frac{H}{2\pi}$ l'integrale $\int fdf.\Psi(f)$ prise depuis f nul, jusqu'à f infini; on aura

$$\int f^4df\varphi(f) = 8\int fdf.\Psi(f) = \frac{4H}{\pi}.$$

Les deux forces tangentielles précédentes parallèles aux axes des x et y deviendront ainsi :

$$(SC+E).H, \quad (3F+D).H.$$

[17, (*Supplément*) p.5]

(Trans. by Bowditch.)

(1) The integral relative to f may be taken from $f = 0$ to $f = \infty$, so that it is independent of the dimensions of the attracting mass. This is what characterizes this kind of attractions, which, being sensible only at insensible distance, allows us to notice or neglect, at pleasure, the attractions of the bodies situated beyond their sphere of sensible activity.

(2) We shall put, as in

$$\Pi(f) = c' - \int_0^f df.\varphi(f),$$

the integral $\int df.\varphi(f)$ being taken from $f = 0$, and c being its value when f is infinite. $\Pi(f)$ will be a positive quantity, decreasing with extreme rapidity; and we shall have, by taking the integrals from $f = 0$;

$$\int f^4df.\varphi(f) = -f^4.\Pi(f) + 4\int f^3df.\Pi(f). \tag{7}$$

$-f^4.\Pi(f)$ is nothing when $f = \infty$; for although f^4 then becomes infinite, the extreme rapidity with which $\Pi(f)$ is supposed to decrease, renders $f^4.\Pi(f)$ nothing.

(3) The functions $\varphi(f)$ and $\Pi(f)$ may be very well compared with exponentials like c^{-if}; c being the number whose hyperbolic logarithm is unity, and i being a very great positive and integral number.

(4) For c^{-if} is finite when $f = 0$, and becomes nothing when f is finite; moreover it decreases with extreme rapidity, and in such a manner that the product $f^n.c^{-if}$ always vanishes when f is infinite, whatever be the value of exponent n.

(5) We shall now put, as in,

$$\int_0^f f df.\Pi(f) = c' - \Psi(f);$$

c' being the value of that integral when f is infinite. $\Psi(f)$ will also be a positive quality decreasing with extreme rapidity; and we shall have

$$4\int f^3 df.\Pi(f) = -4f^2.\Psi(f) + 8\int f df.\Psi(f).$$

When f is infinite, $f^2.\Psi(f)$ becomes nothing; therefore we shall have, by taking the integral fron $f = 0$ to $f = \infty$

$$4\int_0^\infty f^3 df.\Pi(f) = 8\int_0^\infty f df.\Psi(f). \tag{8}$$

(6) Lastly if we put as in,

$$\frac{H}{2\pi} = \int_0^\infty f df.\Psi(f),$$

we shall have,

$$\int_0^\infty f^4 df \varphi(f) = 8\int_0^\infty f df.\Psi(f) = \frac{4H}{\pi}. \tag{9}$$

Thus the two preceding tangential force, parallel to the axes of x and y, will become

$$(SC + E).H, \quad (3F + D).H.$$

[18, pp.812-813]

Remark by us: above (9) tells us simply that we get its equation from (7) and (8),

$$\int_0^\infty f^4 df.\varphi(f) = -f^4.\Pi(f) + 4\int_0^\infty f^3 df.\Pi(f) \cdots (7), \quad 4\int_0^\infty f^3 df.\Pi(f) = 8\int_0^\infty f df.\Psi(f) \cdots (8).$$

¶ 18 page of *Supplement* :

Fixons à cette extrémité, l'origine des coordinées x, y, z d'un point quelconque du plan solide; l'axe des x étant sur la ligne a de la plus courte distance de l'extrémité de la droite au plan, et l'axe des y étant horizontal comme l'axe des x.

En désignant par z' l'abaissement au-dessous de l'origine des coordonées, d'un point quelconque de la ligne attirée; l'attraction vertical du plan solide sur ce point sera à la distance s, et s

$$\iiint dx.dy.dz.\frac{(z + z')}{s}.\varphi(s);$$

$\varphi(s)$ étant la loi de l'attraction à la distance d'un point attirant du plan, au point attiré de la ligne ; ensorte que l'on a

$$s^2 = x^2 + y^2 + (z + z')^2.$$

Pour avoir l'attraction verticale du plan solide, sur la ligne entière; il faut multiplier la triple intégrale précédente par dz', et l'intégrer par rapport à z' depuis $z' = 0$ jusqu'à z' infini.

En désignant donc comme dans le n^o 1 de ma Théorie de l'action capillaire, par $c - \Pi(s)$, l'intégrale $\int ds.\varphi(s)$ prise depuis $s = 0$, la constante c étant l'intégrale entière depuis s nul jusqu'à s infini; on aure

$$\int dz'.\frac{(z + z')}{s}.\varphi(s) = \Pi(s);$$

s étant dans la second membre de cette équation, ce que devient s, à l'origine des coordonnées, ou lorsque z' est nul.

L'attraction du plan solide sur la ligne entière sera donc

$$\iiint dx.dy.dz.\Pi(s).$$

[17, (*Supplément*) pp.18-19].

2.2. Gauss' paper.

2.2.1. Gauss' papers of the capillary action.

Gauss states common motivations with Laplace about MD equations. For example, in §10, §11, §12, which are bellow pages, he states the difficulties of integral $\int r^2 \varphi r.dr$, in which he confesses that he also is included in the person who feels difficulties to calculate the MD integral.

2.2.2. Gauss' letters corresponded with Bessel about Laplace's theory of the capillary action.

Gauss corresponded with Bessel about Laplace's two papers[17].

> Allein in der ganzen ersten Abhandlung selbst finde ich kein Wort, was dienen kann diess zu beweisen. Es kann also wohl nichts gemeint sein als die Stelle in der Einleitung pag. 5, wo ich aber den Schluß, daß die ≫ plans (en question) sont égalment inclinés à leurs parois≪ keineswegs auf eine befriedigende Art begründet finde. Ich gestehe, daß mir dieser Hauptheil von Laplace's Theorie der praecisen mathematischen Begründung des übrigen keineswegs würdig zur Seite zu stehen, sondern mehr den Character der vaguen Aperçus, die man früher von dem granzen Phaenomene hatte, to tragen scheint.
>
> Freilich könnte man sagen, daß Laplace these Lücke einigermassen in der zeiten Abhandlung ausgefüllt hat. Das Rapprochement in der ersten Methode die Haarröhrchen zu behandeln mit der andern in der zweiten Abhandlung (die doch wohl im Grunde nichts weiter ist als die Ladande'sche) führt zu einer Bestimmung des Winkels quaestionis, pag. 18. (27. Januar 1829.) [11, pp.487-490].
>
> (Trans.) Only in all the first paper, I can find no word to be useful for me. It is sufficient to be no meaning as the part of the introduction [9] in page 5, where I conclude that his phrase "the plane (in question) inclines equally to its wall" is not based on the admited method. I can not help confessing that these main theory by Laplace's Theory is for me to be convinced which is never worth to consult it as the (concise) [10] mathematical ground.
>
> Although we can say, of cource, that Laplace complemented these defects in the second paper, however, his approximation in the first method, dealt the capillar action with another one, in the second paper (which is fundametally inferior to the writing by Ladande'sche[11]), he deduces to the doubtful formulae of angle. page 18.

2.2.3. Bessel's reply to Gauss.

> Gegen die Gleichung der Oberfläche habe ich nie ein Misstrauen empfinden, allein den Winkel habe auch ich nicht für erwiesenermaßen unabhängig von dem Durchmesser der Röhre u.s.w. gehalten, sondern diese vielmehr als der Erfahrung, welche mit dem Raisonnement Seite 5 zusammentrifft, entsprechend; denn das Aufsteigen der Flüssigkeit in eigen Röhren könnte nicht dem Durchmesser derselben umgekehrt propotional sein, wenn dieser Winkel nicht stets gleich bleibe. (10. Februar 1829) [11, pp.491-493].

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[9] The introduction takes 1-9 pages in [17] and 685-694 pages in [18].

[10] We do not know about the meaning "praecisen". We can consult the word "praecise" whose meaning is "in short, in few words, briefly, concisely " of only as adverb with the following dictionaries edited by C.T.Lewis, "Elementary Latin Dictionary Lexicon" [20], or "Lexicon Latino-Japonicum" by Kenkyusha. In this sentence by Gauss, it must be used as adjective, so that we use as "concise".

[11] An astronomer who then was criticized for his astronomical writings.

(Trans.) To the equation of surface, I did not have any doubts, however, about that the angle is independent of the diameter of the tube, etc., I have not accepted as being beyond doubt, but also these, strictly speaking, in the experience, which considering with the assumption of the page 5, phenomena of fluid in the tube, it is impossible to be in inverse propotion to the diameter of the tube, because this angle is not always equal.

3. The reserved problems between Navier and others including Poisson, Cauchy, etc. on the molecular actions

3.1. A universal method for the two-constants theory.

In this section, we would like to propose a universal method to describe the kinetic equations that arise in isotropic, linear elasticity. This is outlined as follows:

- The partial differential equations describing waves in elastic solids or flows in elastic fluids are expressed by using one constant or a pair of constants C_1 and C_2 such that:
  for elastic solids: $\frac{\partial^2 \mathbf{u}}{\partial t^2} - (C_1 T_1 + C_2 T_2) = \mathbf{f}$,
  for elastic fluids: $\frac{\partial \mathbf{u}}{\partial t} - (C_1 T_1 + C_2 T_2) + \cdots = \mathbf{f}$,
  where $T_1,\ T_2, \cdots$ are the first kind of tensors or terms constituting our equations. For example, the MDNS equations corresponding to incompressible fluids is composed of the kinetic equation along with the continuity equation and are conventionally written, in modern vector notation, as follows:

$$\frac{\partial \mathbf{u}}{\partial t} - \mu \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f}, \quad \text{div } \mathbf{u} = 0. \tag{10}$$

- C_1 and C_2 are the two coefficients of the two-constants theory, for example, ε and E introduced by Navier, or R and G by Cauchy, k and K by Poisson, ε and $\frac{\varepsilon}{3}$ by Saint-Venant, or μ and $\frac{\mu}{3}$ by Stokes. Moreover, C_1 and C_2 can be expressed in the following form:

$$\begin{cases} C_1 \equiv \mathcal{L} r_1 g_1 S_1, \\ C_2 \equiv \mathcal{L} r_2 g_2 S_2, \end{cases} \quad \begin{cases} S_1 = \iint g_3 \rightarrow C_3, \\ S_2 = \iint g_4 \rightarrow C_4, \end{cases} \quad \Rightarrow \quad \begin{cases} C_1 = C_3 \mathcal{L} r_1 g_1 = \frac{2\pi}{15} \mathcal{L} r_1 g_1, \\ C_2 = C_4 \mathcal{L} r_2 g_2 = \frac{2\pi}{3} \mathcal{L} r_2 g_2. \end{cases}$$

- The two coefficients are expressible in terms of either the operator $\mathcal{L}$ ($\sum_0^\infty$ or $\int_0^\infty$) depending on one's personal preference, [12] where r_1 and r_2 are radial functions related to the radius of the active sphere of the molecules, raised to some power of n for Poisson's and Navier's cases, the relationship between these functions can be expressing by a logarithm with base r such that: $\log_r \frac{r_1}{r_2} = 2$.
- g_1 and g_2 are certain functions which depend on r and are described with attraction &/or repulsion.
- S_1 and S_2 are two expressions which describe the surface of the active unit-sphere centered on a molecule through application of the double integral (or single sum in the case of Poisson's fluid).
- g_3 and g_4 are certain compound spherical harmonic functions to calculate the momentum over the unit sphere.
- C_3 and C_4 are indirectly determined as the common coefficients derived from the invariant tensor. With the exception of Poisson's fluid case, C_3 of C_1 is $\frac{2\pi}{3}$, and C_4 of C_2 is $\frac{2\pi}{15}$, which on computing only the molecules, and which are independent of personal preferences. In Poisson's case, we get the same as above after multiplying by $\frac{1}{4\pi}$. integrals are calculated from the total momentum of the active sphere of the
- The ratio of the two coefficients, including Poisson's case, is an invariant: $\frac{C_3}{C_4} = \frac{1}{5}$.

3.2. Poisson vs. Navier vs. Cauchy.

Some reserved problems on the molecular actions in elastic solid/fluid are :

(1) [Priority :] Navier's anger as one of the *géomètres* [13].
(2) [C_1, C_2 :] Navier's ε and E vs. Poisson's k and K vs. Cauchy's G and R. (cf. Table 1.)
(3) [f_1, f_2 :]

---

[12] At the time, there were heated arguments over Navier ' s integration and Poisson ' s summation.

[13] This means the mathematicians, which is used only in old French. G.Green uses this equivalent word in English such as follows : "This hypothesis, at first advanced by M.Cauchy, has since been adopted by several *philosophers*" , in his paper[12, p.305].

TABLE 1. The expression of the total momentum of molecular actions by Laplace, Navier, Cauchy, Poisson, Saint-Venant & Stokes. (Remark. 6–9 : equilibrium, else : kinetic equation)

no	name	problem	C_1	C_2	C_3	C_4	$\mathcal{L}$	r_1	r_2	g_1	g_2	remark
1	Navier [25]	elastic solid	ε		$\frac{2\pi}{15}$		$\int_0^\infty d\rho$	ρ^4		$f\rho$		ρ : radius
2	Navier fluid [26]	motion of fluid	ε		$\frac{2\pi}{15}$		$\int_0^\infty d\rho$	ρ^4		$f(\rho)$		ρ : radius
				E		$\frac{2\pi}{3}$	$\int_0^\infty d\rho$		ρ^2		$F(\rho)$	
3	Cauchy [5]	system of particles	R		$\frac{2\pi}{15}$		$\int_0^\infty dr$	r^3		$f(r)$		$f(r) \equiv \pm[r\mathrm{f}'(r) - \mathrm{f}(r)]$
				G		$\frac{2\pi}{3}$	$\int_0^\infty dr$		r^3		$\mathrm{f}(r)$	$f(r) \neq \mathrm{f}(r)$
4	Poisson [28]	elastic solid	k		$\frac{2\pi}{15}$		$\sum \frac{1}{\alpha^5}$	r^5		$\frac{d.\frac{1}{r}fr}{dr}$		
				K		$\frac{2\pi}{3}$	$\sum \frac{1}{\alpha^5}$		r^3		fr	
5	Poisson [29]	motion of fluid	k		$\frac{1}{30}$		$\sum \frac{1}{\varepsilon^3}$	r^3		$\frac{d.\frac{1}{r}fr}{dr}$		$C_3 = \frac{1}{4\pi}\frac{2\pi}{15} = \frac{1}{30}$
				K		$\frac{1}{6}$	$\sum \frac{1}{\varepsilon^3}$		r		fr	$C_4 = \frac{1}{4\pi}\frac{2\pi}{3} = \frac{1}{6}$
6	Laplace [18]	capillary action	H		2π		$\int_0^\infty dz$	z		$\Psi(z)$		z : distance
				K		2π	$\int_0^\infty dz$				$\Psi(z)$	cf.§9.3, cf.Gauss[10]
6-2	Rewrited by Poisson[31]		H		$\frac{\pi}{4}\rho^2$		$\int_0^\infty dr$	r^4		φr		[31, pp.14-15]
				K		$\frac{2\pi}{3}\rho^2$	$\int_0^\infty dr$		r^3		φr	
7	Poisson [31]	capillary action	H		$\frac{\pi}{4}\rho^2$		$\int_0^\infty dr$	r^4		φr		[31, p.14]
				K		$\frac{2\pi}{3}\rho^2$	$\int_0^\infty dr$		r^3		φr	[31, p.12]
8	Navier fluid [26]	equiliblium of fluid	p		$\frac{4\pi}{3}$		$\int_0^\infty d\rho$	ρ^3		$f(\rho)$		ρ : radius
9	Poisson [29]	equiliblium of fluid	q		$\frac{1}{4}$		$\sum \frac{1}{\varepsilon^3}$	$\frac{1}{r}$		$r_i^2 z' R$		$C_3 = \frac{1}{4\pi}\pi = \frac{1}{4}$
				p		$\frac{1}{6}$	$\sum \frac{1}{\varepsilon^3}$		r		R	$C_4 = \frac{1}{4\pi}\frac{2\pi}{3} = \frac{1}{6}$
10	Saint-Venant [36]	fluid	ε	$\frac{\varepsilon}{3}$								
11	Stokes [37]	fluid	μ	$\frac{\mu}{3}$								
12	Stokes [37]	elastic solid	A	B								$A = 5B$

- Should we describe by which of *attraction* &/or *repulsion* on the function between two molecules ?
- Navier's $f(\rho)$ and $F(\rho)$ vs. Poisson's fr vs. Cauchy's $\mathrm{f}(r)$ and $f(r)$.
- Navier's $e^{-k\rho}$ for an exponential function as the example of fr vs. Poisson's $ab^{(-\frac{r}{na})^m}$.

(4) [$\mathcal{L}$:] Navier's integral vs. Poisson's summation with mean value of the molecular intervals in the range from 0 to ∞.

(5) [Target of fluid :] Navier's *incompressiblity* vs. Poisson's *compressiblity* (including *incompressiblity*).

3.2.1. Should we describe by which of *attraction* &/or *repulsion* on the function between two molecules ?

TABLE 2. C_1, C_2 and equation of equiliblium of fluid containing exact differential by Poisson & Navier

no	name	C_1, C_2 of equiliblium	equation of equiliblium with extact differential term
1	Poisson [29]	$C_1 = -q \equiv \frac{1}{4\epsilon^3} \sum \frac{r_1^2 z' R}{r}$ $C_2 = p \equiv \frac{1}{6\epsilon^3} \sum rR$	$N = p + q\left(\frac{1}{\lambda} + \frac{1}{\lambda'}\right)$ where N : the vertical force, λ, λ' : the radii of the principal curvature
2	Navier fluid [26]	$C_1 = p \equiv \frac{4\pi}{3} \int_0^\infty d\rho \rho^3 f(\rho)$ $C_3 = \int_0^{\frac{\pi}{2}} d\psi \int_0^{\frac{\pi}{2}} d\varphi g_3$ $\Rightarrow \{\frac{2}{3}, \frac{1}{3}, \frac{\pi}{4}\} \Rightarrow \frac{8\pi}{6} = \frac{4\pi}{3}$	$0 = \iiint dxdydz \left[p\left(\frac{d\delta x}{dx} + \frac{d\delta y}{dy} + \frac{d\delta z}{dz}\right) + P\delta x + Q\delta y + R\delta z\right]$. By partial integral $0 = \iiint dxdydz \left[\left(P - \frac{dp}{dx}\right)\delta x + \left(Q - \frac{dp}{dy}\right)\delta y + \left(R - \frac{dp}{dz}\right)\delta z\right]$ $- \iint dydz\left(p'\delta x' - p''\delta x''\right) - \iint dxdz\left(p'\delta y' - p''\delta y''\right) - \iint dxdy\left(p'\delta z' - p''\delta\right.$ $\Rightarrow$. condition of inner point and exact differential $\frac{dp}{dx} = P, \quad \frac{dp}{dy} = Q, \quad \frac{dp}{dz} = R. \quad \Rightarrow \quad dp = Pdx + Qdy + Rdz$ $\Rightarrow$. boundary condition and relation of variation $\delta x, \delta y, \delta z$ $0 = Pdx + Qdy + Rdz \quad \Rightarrow \quad 0 = \delta x \cos l + \delta y \cos m + \delta z \cos n$

Laplace [14] in 1819 : $\varphi(f) = A(f) - R(f)$, Poisson [15] : $R = Fr - fr$, where $\varphi(f)$ & R of the left hand side : a function depends on distance : f & r between two molecules, $A(f)$ & fr : attraction, $R(f)$ & Fr : repulsion. Navier introduces both $f(\rho)$ and $F(\rho)$ in the other meaning of a function on the culculation in partial momentum and in total momentum, in which Navier mentiones about the relations without showing the defference between the two molecular forces as above, and intensifies only repulsion, as follows :

> The force which brings up between these two molecules depend on the situation of the point M, must be balanced with the pression, which can vary in the various particle of fluid. They depend on the distance ρ, and all the molecular actions, attenuate very rapidly when these distance increase. We call these force by the function $f(\rho)$. (Navier[26, p.392])

Navier poses the question about Poisson's $(r' - r)fr$ which is already appeared in Fourier[8, p.35]. We cite the paragraph on this point by D.H.Arnold, who is the leading researcher of Poisson :

> By being somewhat casual in his selection of an example, Poisson succeeds in exposing himself to the sharp criticism of Navier. Still stinging from the abuse that his research on elasticity had suffered at the hand of Poisson, Navier is quick to point out that such an exponential function must be either always positive or always negative. Hence, he argues, the resultant force between molecules would have to be always atractive or always repulsive. He concludes that the "*nature de la fonction présentée par l'auteur semble donc entièrement incompatible avec la notion d'un corps solide.*" [16]
>
> In spite of Navier's observation, Poisson's general discussion makes it clear that he was thinking of his resultant function as being represented as a difference of two functions of the kind described above. In his "Extrait" discussed above, he actually reduces his function R to the form $R = Fr - fr$, where both F and f are functions having only positive values that become insensible at sensible values for r. Presumably Poisson regarded his thoughts on his subject as being without need of further clarification, as he never bothered to answer Navier's objections. D.H.Arnold [1, VI, p.355]

In relating to last paragraph, Poisson describe :

---

[14] N.Bowditch[18, p.685] comments as follows :

> This theory of capillary attraction was first published by La Place in 1806 ; and in 1807 he gave a supplement. In neither of these works is the repulsive force of the heat of fluid taken into consideration, because he supposed it to be unnecessary. But in 1819 he observed, that this action could be taken into account, by supposing the force $\varphi(f)$ to represent the difference between the attractive force of the particles of the fluid $A(f)$, and the repulsive force of the heat $R(f)$ so that the combined action would be expressed by, $\varphi(f) = A(f) - R(f)$; $\cdots$

[15] Poisson[30, p.73], [29, p.6]

[16] Navier [27, p.101]

Cela posé, appelons m et m' les messes de deux molécules voisines, c et c' leurs quantités de calorique, M et M' leurs centres de gravité, et r la distance MM' ; et considérons l'action exercée par m' sur m, laquelle est égale et contraire à la réaction de m sur m'. Supposons d'abord les dimensions de m et de m' très-petites par rapport à la distance qui les sépare. L'action dont il s'agit se réduira alors à une force unique, dirigée suivant la droite MM', et dont l'intensité sera une fonction de r que nous représenterons par R. En même temps, leur répulsion mutuelle sera propotionnelle au produit de c et c', et leur attraction, au produit de m et m'. En considérant la force R comme positive ou négative, selon qu'elle tendra à augmenter ou à diminuer la distance r, sa valeur sera excés de la répulsion sur l'attraction ; et si l'on suppose que l'attraction réciproque de la matière et du calorique qui retient celui-ci dans chaque molécule s'étent au-dehors, il faudra retrancher de cet excès l'attraction du calorique de m' sur la matière de m, et celle de la matière de m' sur le calorique de m ; lesquelle forces seront propotionnelles, la première au produit mc' et la seconde à $m'c$. De cette manière la valeur complète de R sera

$$R = cc'\gamma - mm'\alpha - mc'\beta - m'c\beta';$$

les coefficiens $\gamma, \alpha, \beta, \beta'$, étant des quantités positive : le premier sera indépendant de la nature de m et de celle de m', le second dépendra de l'une et de l'autre, le troisième ne dépendra que de la nature de m, et la quatrième, de celle de m'.

En réunissant les trois derniers termes de R en un seul, on pourra écrire sa valeur sous cette forme :

$$R = Fr - fr.$$

Chacune des deux fonctions Fr et fr n'aura que des valeurs positives ; ces valeurs décroũtront trés-rapidement et sans alternative, à mesure que la variable r augmentera : elles deviendront insensibles pour toute valeur sensible de r. Poisson [29, p.6, § 2]

4. Laplace's *Supplement*

We show Laplace' calculation in *Supplement* as follows :

$$2\pi.\{1 + (A+B).r\}.\Psi(r).$$

Maintenant, si l'on nomme R le rayon osculateur de la section d ela surface, par un plan passant par les axes des x et des z, et si l'on nomme pareillement R' le rayon osculateur de la section de la surface, par un plan passant par les axes des y et des z ;

$$A = \frac{1}{2R}, \quad B = \frac{1}{2R'}.$$

$$2\pi.\left\{1 + \frac{r}{2}.\left(\frac{1}{R} + \frac{1}{R'}\right)\right\}.\Psi(r).$$

He stated as follows :

Pour avoir l'action entière du corps, sur un fluide renfermé dans un canal infiniment étroit per pendiculaire à la surface, et dont la base est prise pour unité; il faut multiplier l'expression précédente par dr, et l'intégrer depius $r = 0$ jusqu'à r infini. Soit alors [17]

$$2\pi \int \Psi f.df = K, \qquad 2\pi \int \Psi f.f.df = H, \tag{11}$$

l'action du corps sur le canal, sera

$$K + \frac{H}{2}.\left(\frac{1}{R} + \frac{1}{R'}\right);$$

When we denote $h + z$ the height of the point on the sea level, g : mass gravity and D : density, then

$$gD.(h+z) = \frac{H}{2}.\left(\frac{1}{R} + \frac{1}{R'}\right).$$

[17] cf. Gauss cites this Laplace's (11) in (21).

However, if we denote by (2)

$$\frac{dz}{dx} \equiv p, \quad \frac{dz}{dy} \equiv q$$

and by the theory of curving surface :

$$\frac{1}{R} + \frac{1}{R'} = \frac{(1+q^2).\frac{dp}{dx} - pq.\left(\frac{dp}{dy} + \frac{dq}{dx}\right) + (1+p^2).\frac{dq}{dy}}{(1+p^2+q^2)^{\frac{3}{2}}} \tag{12}$$

$$\frac{1}{2}.H.\left[\frac{(1+q^2).\frac{dp}{dx} - pq.\left(\frac{dp}{dy} + \frac{dq}{dx}\right) + (1+p^2).\frac{dq}{dy}}{(1+p^2+q^2)^{\frac{3}{2}}}\right] = gD.(h+z)$$

équation qui est visiblement la mêmé que l'équation (a) [18] du *no* 4 de la Théorie citée.

Maintenant, il est facile de s'assurer par la théorir des surfaces courbes, que si l'on nomme ϖ l'angle que la plan tangent à la surface du fluide intérieur au tube, forme avec les parois du tube toujours supposé vertical, à l'extrémité de sa sphère d'activité sensible ; on a

$$\cos\ \varpi = \pm \frac{pdy - qdx}{ds.\sqrt{1+p^2+q^2}}$$

ds étant l'élément de la section ; on a donc en observant que l'angle ϖ est constant, comme je l'ai fait voir dans la théorie citée,

$$\pm \int \frac{pdy - qdx}{\sqrt{1+p^2+q^2}} = c.\cos\ \varpi$$

c étant le contour entier de la section; partant

$$\frac{1}{2}.H. \iint dxdy.\left\{\left(\frac{d.\frac{p}{\sqrt{1+p^2+q^2}}}{dx}\right) + \left(\frac{d.\frac{q}{\sqrt{1+p^2+q^2}}}{dy}\right)\right\} = \frac{1}{2}.H.c.\cos\varpi$$

ce qui donne

$$gD.V = \frac{1}{2}.H.c.\cos\varpi$$

ainsi le volume du fluide, élevé au-dessus du niveau par l'action capillaire, est proposionnel au contour de la section de la surface intérieure du tube. On peut parvenir à cette équation remarquable, en considérant sous le point de vue suivant, les effets de l'action capillaire.

5. "CHARACTERISTICS" IN THE GAUSS' PAPERS

Poisson says about Gauss[10]: Gauss' success is due to the merit of his *characteristic.*

There are five types of ≺ *"indoles"* ≻ / ≺ *characteristics* ≻ / ≺ *character* ≻ in the Gauss' papers[9, 10] such as :

(1) the ≺ *"indoles"* ≻ / ≺ *characteristics* ≻ as a function
(2) the ≺ *characteristics* ≻ as a force
(3) the ≺ *"indoles"* ≻ / ≺ *characteristics* ≻ as nature
(4) the ≺ *characteristic* ≻ as an operator
(5) the ≺ *character* ≻ as a symbol

Here, the word ≺ *"indoles"* ≻ is the proper Latin, according to a Latin dictionary : Lewis[20], *"indoles"* means : an inborn quality, natural quality, nature (sic. in English).

TABLE 3. usage of *"indoles"/characteristic/character* in Gauss[9, 10]

no	use	*"indoles"*	*characteristic*	*character*
(1)	function	1-1([10], preface)	1-2([10], §6)	
(2)	force		2-1([10], §2), 2-2([10], §2), 2-3,2-4([10], §18)	
(3)	nature	3-1([10], §4), 3-2([10], §14), 3-3([10], §20)	3-1([10], §4)	
(4)	operator		4-1([10], §2)	
(5)	symbol			5-1([9], §21), 5-2([9], §22), 5-3([10], §8), 5-4([10], §10)

We distinguish these $\prec$ *"indoles"* $\succ$ /$\prec$ *characteristics* $\succ$ / $\prec$*characters* $\succ$ in detail of the examples in Gauss [9, 10] as follows :

(1) the $\prec$ *"indoles"* $\succ$ / $\prec$ *characteristics* $\succ$ as a function

•(1-1) ([10], preface) In a word, the $\prec$ *indoles* $\succ$ of the function φf is reserved ineffective, as long as f were an arbitrary, infinitesimal value.

> ⇒ $\prec$ *Indoles* $\succ$ functionis φf prorsus intacta linquitur, dummode insensibilis sit pro omnibus valorribus sensibilibus ipsus f. $\cdots$ [10, 32],

•(1-2) ([10], §6) In this survey, we denote the spaces by s and S, the function on distance denoted with the $\prec$ *characteristic* φ $\succ$.

> ⇒ Spatia in hac disquisitione generali par s, S, functionem distantiae per $\prec$ *characteristicam* φ $\succ$ denotabimus, $\cdots$ [10, 39],

(2) The $\prec$ *characteristic* $\succ$ as a force

•(2-1) ([10], §2) II. The attractive force, which itself coresponds to the points $m, m', m'', \cdots$. The intensity of attraction of function is propotional with the distance if this function, the $\prec$ *characteristic* $\succ$ denoted by f in mass and supposed that the attraction is uniformly concentrated in the point.

> ⇒ II. Vires attractivae, quas puncta $m, m', m'', \cdots$ a se mutuo experiuntur. Intensitas attractuionis function distantiae propotionalis sive producto huius functionis per $\prec$ *characteristicam* f $\succ$ denotandae in massam in puncto attrahente concentratam aequalis supponitur. [10, 36],

•(2-2) ([10], §2) III. The forces, $m, m', m'', \cdots$ are attractive to the infinitesimal fixed points. For these forces, with the similar way, we will designate the *characteristic* F such that the inverse-directional distance is used, [19] and with $M, M', M'', \cdots$, which are treated as a fixed point in one case, or a mass in the other case, which are supposed in these concentrate.

> ⇒ III. Vires, quibus puncta $m, m', m'', \cdots$ ad puncta quotcunque fixa attrahuntur. Pro his viribus simili mode $\prec$ *characteristica* F $\succ$ distantiae praefigenda utemur, et per $M, M', M'', \cdots$ tum puncta fixa, tum massas, quae in ipsis concentratae supponuntur, designabimus. [10, 36],

•(2-3),(2-4) ([10], §18)

- In the evolution of the third term in the expression of Ω (See (25),) exist keeping the symbol S, by which we denote the space filled in the vase,
- and we put the $\prec$ *characteristic* F $\succ$ of the $\prec$ *characteristic* f $\succ$ as the attractive force of the molecule of vase which is capable to substitute its relation,

[18] cf. (5).

[19] In Gauss' paper there is no word of "the repulsive force" of the heat, which makes the pair of mutual action. But he uses two types f and F, and F is the inverse distance to f. (Latin is : Pro his viribus simili mode $\prec$ *characteristica* F $\succ$ distantiae praefigenda utemur.) See the origin of Latin in (2-2).

- and as the same way, we put the functions by the ≺ *characteristic* $\varphi, \psi, \theta, \theta'$ ≻ denoting the same one by $\Phi, \Psi, \Theta, \Theta'$ denoted to apply complying to depend on the relation between f and F.

  ⇒ In evolutione termini tertii expressionis Ω signum S retinendum erit, ut denotet spatium a vase repletum, sed loco ≺ *characteristicae* f ≻ , ≺ *characteristicam* F ≻ ad vim attractivam molecularum vasis relatam substituere oportebit, et perinde loco functionum per ≺ *characteristicas* $\varphi, \phi, \theta, \theta'$ ≻ denotatarum alias per ≺ *characteristicas* $\Phi, \Psi, \Theta, \Theta'$ ≻ denotandas adhibere, quas perinde ab F pendere supponimus ut illas ab f. ⋯ [10, 53-54],

(3) The ≺ *"indoles"* ≻ / ≺ *characteristic* ≻ as nature

•(3-1) ([10], §4) The ≺ *characteristics, indoles* ≻ of fluid consists of the perfect mobility, for example, in the minimum particles, however the figure were big, it can be induced to any size, or minimum potential, the mutual figure depends on the changing mutually.

⇒ Corporum fluidorum ≺ *indoles characteristica* ≻ consistit in perfecta mobilitate vel minimarum partium, in quas transit, dum deinceps m cum m', m'', m''' etc. ⋯ [10, 38],

•(3-2)([10], §14) Moreover, that comes from this ≺ "*indole*" function : θ ≻ with respect to the integral (I)

$$\text{integral (I)} : \quad \iint \frac{dt.dT \cos q. \cos Q. \theta(dt, dT)}{(dt, dT)^2}$$

follows and we would like to investigate it.

•(3-3)([10], §20) Moreover, now, with theorem in art.18, we would like to determine the ≺ "*indoles*" ≻ (nature) of the figure in equilibre, these problem are changed in evolution of the general variation, expressed with W, if the motion of the figure of the space filled with a fluid occured in only infinitesimally small size.

(4) The ≺ *characteristic* ≻ as an operator

•(4-1) ([10], §2) The ≺ *characteristic* Σ ≻ represents the expression of sum, in which $m', m'', m''', \cdots$ follow permuting following m.

⇒ ubi ≺ *characteristica* Σ ≻ representat aggregatum expressionis adscriptae cum omnibus, in quas transit, dum deincept m cum m', m'', m''', etc. permutatur. ⋯ [10, 37],

(5) The ≺ *character* ≻ as a symbol

•(5-1) ([9], §21) We would like to restore the general meanings to the ≺ *characters* p, q, E, F, G, ω ≻ , which were accepted, additionally speaking, which are determinated by the dual and alias variables p', q', where, the elements are explained with linear indefinite by : $\sqrt{E'dp'^2 + 2F'dp'.dq' + G'dq'^2}$.

•(5-2) ([9], §22) The general survey in the previous article, we traced to the application of late, where p and q are put with the most general meaning, for p', q', which are adopted in the article 15, in which these ≺ *characters* ≻ are denoted with r and φ.

•(5-3) ([10], §10) Without theory and the policy to investigate that the gravity comes from the hypothesis, in the other point, the law of the function fr, as the same as the unknown problems in general, which we can not help making a mistake about the mathematical ≺ *character* ≻, seem peculier : namely, as long as even the fact, standing on the most precise mathematics, can not punish himself, if so, so much as the mathematical precision, more, even without the experiments, we can get the absolute level of value ; without an experiment (or proof), no one is free from the amusement by oneself in seeking after absolute truth ; if you would like to success, withdrow supposition itself.

6. *Disquisitiones generales circa superficies curvas.* *(General survey on the curved surface)*

We show the only §21 and §22 of the deduction of first and second formulae.

6.1. Deduction of formulae.

We would like to restore the general meanings to the ≺ *characters* p, q, E, F, G, ω ≻ , which were accepted, additionally speaking, which are determinated by dual alias variables p', q', where, the elements are explained with linear indefinite by :

$$\sqrt{E'dp'^2 + 2F'dp'.dq' + G'dq'^2}$$

$$\begin{cases} dp' = \alpha dp + \beta dq, \\ dq' = \gamma dp + \delta dq \end{cases} \Rightarrow \begin{bmatrix} dp' \\ dq' \end{bmatrix} = \begin{bmatrix} \alpha\ \beta \\ \gamma\ \delta \end{bmatrix} \begin{bmatrix} dp \\ dq \end{bmatrix}$$

Now we would like to investigate the geometric meaning of these coefficients $\alpha, \beta, \gamma, \delta$.

Quatuor is now the linear system considered in the surface curve, for these, they were constants such as q, p, q', p'. If we determine these by points, these respond to the variable values of q, p, q', p', the positive variations dq, dp, dq', dp' are responded

$$\sqrt{E}.dp, \quad \sqrt{G}.dq, \quad \sqrt{E'}.dp', \quad \sqrt{G'}.dq'$$

We denote the angles with M, N, M', N'

$$p + dp, \quad q + dq, \quad p' + dp', \quad q' + dp'$$

are independent of the values of variations dq, dp, dq', dp'

$$\sqrt{E}.dp.\sin M + \sqrt{G}.dq.\sin N = \sqrt{E'}.dp'.\sin M' + \sqrt{G'}.dq'.\sin N'$$

We, however, introduce these by notating

- $N - M = \omega$
- $N' - M' = \omega'$
- $N - M' = \psi$.

These equations of the invented methods are seen in the following formats

$$\sqrt{E}.dp.\sin \underbrace{(M' - \omega + \psi)}_{M} + \sqrt{G}.dq.\sin \underbrace{(M' + \psi)}_{N} = \sqrt{E'}.dp'.\sin M' + \sqrt{G'}.dq'.\sin \underbrace{(M' + \omega')}_{N'}, \tag{13}$$

or

$$\sqrt{E}.dp.\sin(\underbrace{N' - \omega}_{M'} - \omega' + \psi) + \sqrt{G}.dq.\sin \underbrace{(N' - \omega' + \psi)}_{M'+N-M'=N} = \sqrt{E'}.dp'.\sin \underbrace{(N' - \omega')}_{M'} + \sqrt{G'}.dq'.\sin N' \tag{14}$$

$$\sqrt{E'}.\sin\ \omega'.dp' = \sqrt{E}.\sin(\omega + \omega' - \psi).dp + \sqrt{G}.\sin(\omega' - \psi).dq \tag{15}$$

$$\sqrt{G'}.\sin\ \omega'.dq' = \sqrt{E}.\sin(\psi - \omega').dp + \sqrt{G}.\sin\ \psi.dq \tag{16}$$

We can construct the equations combinating the left hand-side of (15) with that made by substituting $N' = 0$ in the left hand-side of (14). And also the left hand-side of (16) with that made by substituting $M' = 0$ in the left hand-side of (13) then

$$\begin{cases} \sqrt{E'}.\sin\ \omega'.dp' = \sqrt{E}.dp.\sin(-\omega - \omega' + \psi) + \sqrt{G}.dq.\sin(-\omega' + \psi), \\ \sqrt{G'}.\sin\ \omega'.dq' = \sqrt{E}.dp.\sin(-\omega + \psi) + \sqrt{G}.dq.\sin(\psi) \end{cases}$$

That is

$$\begin{bmatrix} \sqrt{E'}.\sin\ \omega'.dp' \\ \sqrt{G'}.\sin\ \omega'.dq' \end{bmatrix} = \begin{bmatrix} \sqrt{E}.\sin(\omega + \omega' - \psi) & \sqrt{G}.\sin(\omega' - \psi) \\ \sqrt{E}.\sin(\psi - \omega') & \sqrt{G}.\sin\ \psi \end{bmatrix} \begin{bmatrix} dp \\ dq \end{bmatrix} \Rightarrow \begin{cases} dp' = \alpha dp + \beta dq, \\ dq' = \gamma dp + \delta dq \end{cases}$$

$$\begin{cases} \alpha = \sqrt{\frac{E}{E'}}.\frac{\sin(\omega+\omega'-\psi)}{\sin\omega'}, \\ \beta = \sqrt{\frac{G}{E'}}.\frac{\sin(\omega'-\psi)}{\sin\omega'}, \\ \gamma = \sqrt{\frac{E}{G'}}.\frac{\sin(\psi-\omega')}{\sin\omega'} \\ \delta = \sqrt{\frac{G}{G'}}.\frac{\sin\ \psi}{\sin\ \omega'} \end{cases} \tag{17}$$

$$\begin{cases} \cos\ \omega = \frac{F}{\sqrt{EG}}, \\ \cos\ \omega' = \frac{F'}{\sqrt{E'G'}}, \\ \sin\ \omega = \sqrt{\frac{FG-F^2}{EG}}, \\ \sin\ \omega' = \sqrt{\frac{F'G'-F'F'}{E'G'}} \end{cases}$$

$$\begin{cases} \alpha\sqrt{(E'G'-F'F')} = \sqrt{EG'}.\sin(\omega+\omega'-\psi), \\ \beta\sqrt{(E'G'-F'F')} = \sqrt{GG'}.\sin(\omega'-\psi), \\ \gamma\sqrt{(E'G'-F'F')} = \sqrt{EE'}.\sin(\psi-\omega), \\ \delta\sqrt{(E'G'-F'F')} = \sqrt{GE'}.\sin\ \psi \end{cases}$$

Substitution

$$Edp'^2 + 2F'dp'dq' + Gdq'^2$$

by

$$\begin{cases} dp' = \alpha dp + \beta dq, \\ dq' = \gamma dp + \delta dq \end{cases}$$

to

$$Edp^2 + 2Fdpdq + Gdq^2$$

then

$$\begin{cases} E'\left(\alpha dp + \beta dq\right)^2 + 2F'\left(\alpha dp + \beta dq\right)\left(\gamma dp + \delta dq\right) + G'\left(\gamma dp + \delta dq\right)^2, \\ Edp^2 + 2Fdpdq + Gdq^2 \end{cases}$$

$$EG - F^2 = (E'G' - F'F')(\alpha\delta - \beta\gamma)^2$$

$$\begin{cases} (\alpha\delta-\beta\gamma)dp = \delta dp' - \beta dq', \\ (\alpha\delta-\beta\gamma)dq = -\gamma dp' + \alpha dq' \end{cases} \Rightarrow (\alpha\delta-\beta\gamma)\begin{bmatrix} dp \\ dq \end{bmatrix} = \begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix}\begin{bmatrix} dp' \\ dq' \end{bmatrix}$$

$$\begin{cases} E\delta^2 - 2F\gamma\delta + G\gamma^2 = \frac{EG-F^2}{E'G'-F'F'}.E', \\ E\beta\delta - F(\alpha\delta+\beta\gamma) + G\alpha\gamma = -\frac{EG-F^2}{E'G'-F'F'}, \\ E\beta^2 - 2F\alpha\beta + G\alpha^2 = \frac{EG-F^2}{E'G'-F'F'}.G' \end{cases}$$

6.2. First Fundamental Theorem and Second Fundamental Theorem.

The general survey in the previous article, we traced to the application of late, where p, q are put with the most general meaning, for p', q', adopted in the articla 15, in which these $\prec$ *characters* $\succ$ were denoted with r, φ. We put $E' = 1, F' = 0, \omega' = \frac{\pi}{2}, \sqrt{G'} = m$, then from (17) we get as follows :

$$\begin{cases} \alpha = \sqrt{E}.\cos(\omega-\psi), \\ \beta = \sqrt{G}.\cos\psi, \\ m.\gamma = \sqrt{E}.\sin(\psi-\omega), \\ m.\delta = \sqrt{G}.\sin\ \psi \end{cases}$$

Here we show the four equations in the above article, replacing for $\alpha,\ \beta,\ \gamma,\ \delta$, then

$$\begin{cases} \sqrt{E}.\cos(\omega-\psi) = \frac{dr}{dp} \\ \sqrt{G}.\cos\ \psi = \frac{dr}{dq} \\ \sqrt{E}.\sin(\psi-\omega) = m.\frac{dr}{dp} \\ \sqrt{G}.\sin\ \psi = m.\frac{dr}{dq} \end{cases} \tag{18}$$

$$EG - F^2 = E(\frac{dr}{dq})^2 - 2F.\frac{dr}{dp}.\frac{dr}{dq} + G(\frac{dr}{dp})^2 \tag{19}$$

$$(E.\frac{dr}{dq} - F.\frac{dr}{dp}).\frac{d\varphi}{dq} = (F.\frac{dr}{dq} - G.\frac{dr}{dp}).\frac{d\varphi}{dp} \tag{20}$$

7. *Principia generalia theoriae figurae fluidrum in statu aequilibrii.* *(General principles of theory on fluid figure in equilibrium state)*

In this dissertation, Gauss treats many important topics in the modern mathematics.

- Preface
- (§1-§5) Introduction
- (§6-§9) Reduction from the sextuplex integral to the quadruplex integral
- (§10-§12) Criticism of Laplace's molecular calculus of capillarity equations
- (§13-§17) Ideas by Gauss
- (§18-§19) Solving variation problem
- (§20-§24) Deduction of Gauss' integral formula
- (§25-§26) Geometric meaning of curvature ($\frac{d\xi}{dx} + \frac{d\eta}{dy}$ in V)
- (§27-§30) Application of his formula to meniscus
- (§31-§33) Attraction in condition by A, α, β
- (§34) Summary

[20]

7.0. Preface.

¶ 2.

• Since Mr. Laplace, from here, presented conveniently the unique supposition about the inner, molecular activity, moreover, giving up diminution of law for the increasing distance, we have got the first result in the surface of the fluid figure based on the accurate calculus, and have established the general equation for the equilibratory figure, not only the pricise capillary phenomenon as described, but also try to explain the relating problems.

• This investigation is discussed getting the consented with and confirmed in everywhere, by the exact experiment, among the first class of increasing natural philosophers, geometricians, and refrred and criticized by the some authorities from all the directions to the maximum part such as a minor or nonsence.

¶ 3.

• In the calculas by Mr. Laplace, we have at least a thing, which we can give evidence about it, and for which we would not absolutely consent with him.

• In the previous commentary : ≺ *Théorie de l'action capillaire* ≻, denoting with φf intensity of the attraction in the distance f, the integrals [21]

$$\int_x^\infty \varphi f.df = \Pi x, \qquad \int_x^\infty \Pi f.fdf = \Psi x,$$

; The integral of two values : [22]

$$2\pi \int_0^\infty \Psi f.df = K, \qquad 2\pi \int_0^\infty \Psi f.f.df = H, \tag{21}$$

where denoting by π the $\frac{1}{2}$ of the circumference of the circle with radius $= 1$.

[20]We entitled for explanation of contents in each subsection below, where, there was not at all name of title but only the number in Gauss' paper. The section number is the same as Gauss' numbering.

[21]cf. Laplace states the two-constants (11) in his original papaer. Poisson cites these equations in (65).

[22]Poisson rewrite these equations to the equivalent with Laplace. cf, (66), (67).

• In a word, the ≺ *indoles* ≻ of the function φf reserves ineffective, as long as this f were insensible for all sensible value. Hence,

- from only this supposition, it does not deduced absolutely,
- moreover, Πf and Ψf are for the finite values, this function f needs to be infinitesimal, can not absolutely be true, $2\pi \int_0^{\text{finite}} \Psi f.df$ and $2\pi \int_0^{\text{finite}} \Psi f.f.df$ turn into another infinitesimal value of K and H as we read in the dissertation ;
- of cource, the form of function φf may be infinitely imaginary, although the fundamental supposition satisfy, these would be erroneous conclusions for this.
- If it is supposed that φf is complete attraction, the partial form $\frac{a}{f^2}$, depend on the general attraction ;
- but as long as we can not measure the attractive particle, even we know the occurrence in experiment, it is too infinitesimal in comparison with total earth, then if extended infinitely, inferred function Ψf is ristricted to be infinite.

¶ 4.

• However, something similar to simple carelessness form the basis, such that he discusses about the form than about the relating action with it.
• Judging from the second dissertation : ≺ *Supplément à la théorie de l'action capillaire* ≻, Mr. Laplace investigated a little, not only the complete attraction, but also the partial one by φf, and tacitly understood incompletely the general attraction ; by the way, if we would refer the latter by him about our sensible modification, it is easy to see to be conspicuous about it.
• He considers exponential e^{-if} as an example of equivalent function with φf, denoting the large quantity by i, or $\frac{1}{i}$ becomes infinitesimal.
But it is not at all necessary to limit the generality by such a large quantity, the things is more clear than words, we would see fasilest, only to investigate if these integrations would be extend, not only infinite but also to an arbitrary sensible distance, or if anything, occurring wider in the finitely measurable distance in experiment.

¶ 5.

• On the otherhand, a person studied this theory with more decisively mistakes, and to this theory, nobody critisize this sophist. Both are clearly a part owner of it.

• Here we established the general equation for fluid of liberal surface with differential by the partial coordinates : this equation depends on the force by molecular attraction, which the particles of the fluid are in motion, and additionary, this theory is absolute and is never rested essentially deficient in it.
• In addition to this equation between partial differential, (its integration, if it were posturated in analysis, an arbitrary function is induced) it is not sufficient for the figure of surface, determined from all aspects, unless the new conditions were accepted the nature of the fluid in the defined boundary.
• Total condition is set up by another theory, which is, the angle of the plane to the surface of the liberal fluid in tangently contact with the vase (exactly speaking, in the boundary of the sensibly attractive force to the wall of vase) with the plane of the wall of vase, it is a tangential constant, we put with the relation with intensity of the molecular force determinated between vase and fluid, so that, the continuity of figure at the neighbourhood of the contacted with the liberal surface of the fluid is not interrupted.

• Hence, to the cardinal proposition of the total theory with calculation for demonstration, we can not accept the papers by Mr. Laplace ; in p.5, since not only he developed clearly incorrect argument but also showed the false proofs : we consider that calulations in the pages in and the followings after p.44 [a] have *non effect in vain.*

[a]There are 35 pages of calculation between p.44 and p.78 in his *Supplement.*

[11, p.33-34]

7.1. Introduction.

On the stabilizing the equation of equilibrium of the system of physical point, we would like to clearify how many which motion confine the condition, provided that the principle of motion of force adapts at maximum.

We woulld like to construct the system as follows :

- from the physical point $m,\ m',\ m'',\cdots$, in which we denote the mass of the concentrate [23] by this letter, we think, which is accepted,
- we figure that
  - P is a from the accerarated force which is active in point m, and these systems of motion, made by an arbitrary material, infinitesimally small, recognize the condition of the affinity of system (motion of force),
  - dp is the motion of the point m in direction of the projection of force P, i.e., made by the angle of cosine, which face with the direction of the force P, multiplicated ;
- next, $\sum Pdp$ is the production of the sum of all similar one with respect to all force of the sole point m.
- As the same way, P' represents the indefinite force of the sole point m',
- in addition to, dp' is the motion of the point m' made with the projection of singular direction, similary with the other points.

From these idea, the condition of equilibrium of the system is consisted of that and the sum are

$$m\sum Pdp + m'\sum Pdp' + m''\sum Pdp'' + \cdots$$

for anywhere, the force of motion becomes = 0, where, the principle motion of the general force is explained, such as precise, in them, and from this sum for null motion, we can get the positive value.

7.2. Three capitals of force.

We consider the force reducing to three capitals.

- I. Gravity.
- II. The attractive force, which itself coresponds to the points $m, m', m'', \cdots$. The intensity of attraction of function is propotional with the distance if this function, the ≺ *characteristic* ≻ denoted by f in mass and supposed that the attraction is uniformly concentrated in the point.
- III. The forces, $m, m', m'', \cdots$ are attractive to the infinitesimal fixed points. For these forces, with the similar way, we will designate the ≺ *characteristic F* ≻ such that the inverse-directional distance is used, and with $M, M', M'', \cdots$, which are treated as a fixed point in one case, or a mass in the other case, which are supposed in these concentrate.

We get $\sum Pdp$ of the previous article as follows :

$$\begin{aligned} &-gdz \\ &- \quad m'f(m,m')d(m,m') - m''f(m,m'')d(m,m'') - m'''f(m,m''')d(m,m''') - \cdots \\ &- \quad MF(m,M)d(m,M) - M'F(m,M')d(m,M') - M''F(m,M'')d(m,M'') - \cdots \end{aligned} \tag{22}$$

where, the difference $d(m,m')$, $d(m,m'')$ etc. are partial, relative to the only motion of the force of m. We denote φ such that :

$$-fx.dx = d\varphi x, \qquad \int fx.dx = -\varphi x, \tag{23}$$

[23]In this paper, Gauss cites about the concentrate in §2, 18.

And by the same way, on Φ,

$$-Fx.dx = d\Phi x, \qquad \int Fx.dx \equiv -\Phi x \tag{24}$$

then we get the integral of it from (22) as follows :

$$\begin{aligned} & -gdz \\ & \quad + \quad m'd\varphi(m,m') + m''d\varphi(m,m'') + m'''d\varphi(m,m''') + \cdots \\ & \quad + \quad Md\Phi(m,M) + M'd\Phi(m,M') + M''d\Phi(m,M'') + \cdots \end{aligned}$$

$$\begin{aligned} \Omega = & -gmz - gm'z' - gm''z'' - gm'''z''' - \cdots \\ & + \quad m\Big\{m'\varphi(m,m') + m''\varphi(m,m'') + m'''\varphi(m,m''') + \cdots\Big\} \\ & + \quad m'\Big\{ \qquad m''\varphi(m,m'') + m'''\varphi(m,m''') + \cdots\Big\} \\ & + \quad m''\Big\{ \qquad m'''\varphi(m,m''') + \cdots\Big\} \\ & + \quad \cdots \\ & + \quad m\Big\{M\Phi(m,M) + M'\Phi(m,M') + M''\Phi(m,M'') + \cdots\Big\} \\ & + \quad m'\Big\{M\Phi(m',M) + M'\Phi(m',M') + M''\Phi(m',M'') + \cdots\Big\} \\ & + \quad m''\Big\{M\Phi(m'',M) + M'\Phi(m'',M') + M''\Phi(m'',M'') + \cdots\Big\} \\ & + \quad \cdots \end{aligned}$$

The function Ω is expressed by the following sequence :

$$\begin{aligned} \Omega = \sum m\{-gz + \frac{1}{2}m'\varphi(m,m') + \frac{1}{2}m''\varphi(m,m'') + \frac{1}{2}m'''\varphi(m,m''') + \cdots \\ +M\Phi(m,M) + M'\Phi(m,M') + M''\Phi(m,M'') + \cdots \end{aligned}$$

where, $\prec$ *characteristic* $\Sigma \succ$ represents the expression of sum, in which $m', m'', m''', \cdots$ follow permuting after m.

7.3. The sum of force : Ω.

If we locate the discreate points $M, M', M'', \cdots$, and assume the continuous corpus extending in the space S, and C is the uniformalized density, then the sum

$$M\Phi(m,M) + M'\Phi(m,M') + M''\Phi(m,M'') + \cdots$$

is transformed into the integral

$$C\int dS.\Phi(m,dS)$$

in the total space S, in which we denote the second analogy with (m, dS), which means the distance from the point m to the arbitrary points in the space S, and we call its element dS.

In addition, if we locate the discreate points $m, m', m'', \cdots$, and assume the continuous corpus extending in the space s, and the uniformal dencity is c, then we get the sum :

$$-gz + \frac{1}{2}c\int ds.\varphi(\mu, ds) + C\int dS.\Phi(\mu, dS)$$

where, z is the altitude of the point μ in the superplane H, in addition, we integralate the first integral, over the total space s and the second integral, over the space S.

$$\Omega = c\int ds.[ds]$$

integrate over the total space s. For brevity, we express :

$$\Omega = -gc\int zds + \frac{1}{2}c^2 \iint ds.ds'.\varphi(ds, ds') + cC\iint ds.dS.\Phi(ds, dS) \tag{25}$$

where, s, s' are specially denoted spaces (satisfied with the mobile material), however with the duplex integration, integrate twice with the element to resolve it.

7.4. The *characteristics*, *indoles* of fluid.

The ≺ *characteristics* ≻ , ≺ *indoles* ≻ of fluid consists of the perfect mobility, for example, in the minimum particles, however the figure were big, it can be induced to any size, or minimum potential, the mutual figure depends on each changing. In unexpansible fluid (the liquid), which we called in our discussion, the volume of this particle keep to be constant due to the all movable figure. Consider that the following motion of this fluid

- which is limited by the solid corpus (the vase),
- and which are obeyed by the attraction between the mutual paticles,
- the attraction between the particles of fluid,
- and the attraction between the particle of fluid and that of the vase,
- the status of equilibrium,
- and value of this Ω, when Ω is maximum, etc.
- and without infinite transpotation between the particle of fluid, this Ω can induce positive increment.

Why this Ω can get the value, as long as such as :

- how long the period the figure,
- what sort of fluid satisfy it,
- moved (only by the interior fluid),
- accepting the equilibrium,
- how many times Ω for zero bring up the infinitesimal motion with the figure of vase.

!pl !dv

Therefore, here, we consider that, if we can assume the figure does not move at all, (the vase which the fluid is contained, is along and tangential in everywhere), the force can not move in the fluid the interior of the fluid, if the equiliblium is holds by itself.

7.5. The expression of Ω : the fundamental theory of fluid equilibrium.

We would like to proceed to precisely investigate the expression of Ω, which we must consider as if the fundamental of the theory of fluid equilibrium.

⇒ Progredimur ad accuratiorem investigationem expressionis Ω, quae tamquam fundamentum theoriae aequilibrii floidorum considerari debet.

We would like to take up, at first,

- the term $\int zds$: the production made by the volume of the space s at the altitude of the central gravity of the surface plane H.
- In addition, $gc\int zds$: the production of mass at the altitude of the fluid.

Hence, thus fluid particles does not obey the other force except for the gravity, in the state of equilibrium, the center altitude of the gravity becomes minimum as possible as, and therefore, we get easy the liberal part or liberal parts of surface, the part of the horizontal plane in the one same place, it becomes the surface and boundary of fluid.

7.6. Transformation of the expression and the definition of s, S, φ, Φ.

We take the transformation as follows :

- of the second and third terms to two cases of the paticular problem, where, propotition of the dual spaces whatever, single element of the first space with second element, we combine and product from the third factor, put from the element volume of the first space and the volume element of the second space, and the function data of the mutual distance, and then we can sum up to the last,

- the second term to the same way, where the both space is the same,
- the third to it, where all of a side of space is from the other side of space : the problem is completed.

the dual different cases is completed, clearly

- when one side of space is part of the other side of space,
- or when have the common part of the other with the other part

Althought, moreover, the first case is sufficient to institute us, or we can easy return the rest to the other side, when the work evaluate, the problem in itself complete by accepting the gerneral sign.

In this survey, we denote the spaces by s and S, the function on distance denoted with the $\prec$ *characteristic* $\varphi \succ$, as the same as in the application to the second located term S and s, in the application to the third located term φ in replacing to Φ. The integration is given as follows :

$$\iint ds.dS.\varphi(ds, dS) \tag{26}$$

We would like to show that the spacial elements, depending on the three variables, which imply that the sextuplex integral are to be reduced to the quadruplex integral.

7.7. Preparation to evolute the equation.

$$\int ds.\varphi(\mu, ds)$$

where μ is the fixed point in the exterior or interior of the space s. We consider the suface of sphere with radius = 1 of which the center is μ.

$$d\Pi = \pm\frac{dt'.\cos q'}{r'r'} = \pm\frac{dt''.\cos q''}{r''r''} = \pm\frac{dt'''.\cos q'''}{r'''r'''} \quad \text{etc.}$$

$$\int r^2\varphi r.dr = -\varphi r$$

We integrate :

$$\int ds.\varphi(\mu, ds)$$

where μ is the fixed point in the exterior or interior of the space s

$$\int ds.\varphi(\mu, ds) = d\Pi.(\psi r' - \psi r'' + \psi r''' + \text{ etc }) = \frac{dt'.\cos q'.\psi r'}{r'r'} + \frac{dt''.\cos q''.\psi r''}{r''r''} + \frac{dt'''.\cos q'''.\psi r'''}{r'''r'''} + \cdots$$

at the time when μ exists in the exterior of the space s :

$$\begin{aligned}
&\int ds.\varphi(\mu, ds) \\
&= d\Pi.(\psi 0 - \psi r' + \psi r'' - \psi r''' + \text{ etc }) \\
&= d\Pi.\psi 0 + \frac{dt'.\cos q'.\psi r'}{r'r'} + \frac{dt''.\cos q''.\psi r''}{r''r''} + \frac{dt'''.\cos q'''.\psi r'''}{r'''r'''} + \cdots
\end{aligned}$$

at the time when μ exists in the interior of the space s.

When we take the sum by the arbitrary surface of the spherical part, we get the integral $\int ds.\varphi(\mu, ds)$ is completed, then

$$\int ds.\varphi(\mu, ds) = \begin{cases} \frac{dt.\cos q.\psi r}{r^2} & \text{in the first case} \\ 4\pi\psi 0 + \frac{dt.\cos q.\psi r}{r^2} & \text{in the second case} \end{cases}$$

where

- dt : the infinite arbitrary elements on the surface of space s,
- q, r : these are the values underlined in the previous pages about the determinate expressions, with respect to the element of r,
- π : $\frac{1}{2}$ of the circumference of circle with its radius = 1.

We see easy the rest, if the point μ is neither interior, nore exterior of the space s, or in the surface of these, to satisfy the secondary formula, the factor will move 4π in 2π, even if the surface in the point μ were offered neither as the cusp nore as the aciform [24] type ; however, by our proposition, it is not at all necessary to satisfy this case.

7.8. Evolution of equation $\iint ds.dS.\varphi(ds, dS)$.

By the discussion in the previous article, the evolution of equation $\iint ds.dS.\varphi(ds, dS)$ reduced to

$$4\pi\sigma\psi_0 + \iint dt.dS.\frac{\cos\ q.\psi(dt, dS)}{(dt, dS)^2}$$

where σ denotes volumes of these spaces, is common in both space s, S, if s, S alternate mutually, the first term $4\pi\sigma\psi_0$ vanishes. New integral seems duplex in external form, but it turns to quintuplex. When we reduce to the quadruplex, we must consider the integral

$$\int dS.\frac{\cos\ q.\psi(\mu, dS)}{(\mu, dS)^2}$$

by the arbitrary elements of the space S are extended, denoting again μ fixed point, and q : angle inter two rectangles ($0 \leq q \leq \pi$) proficient point. Others are easily perspective, if the point μ is only exterior or interior of the space s, evaluate the secondary formula, move the factor 4π to 2π, and then if our propositions are not useful for you, please read the following cases.

$$d\Pi = -dT''.\cos\chi' = dT'''.\cos\chi'' = -dT''''.\cos\chi''' \quad \text{etc.}$$

$$\int \frac{dr.\psi r}{r^2} = -\theta r$$

here, accepting arbitrary the integral constant, our integral of the interior space S of prism,

$$\begin{aligned} &= d\Pi.(\theta R' - \theta R'' + \theta R''' - \quad \text{etc}) \\ &= -dT'.\cos\chi'\theta R' - dT''.\cos\chi''\theta R'' - dT'''.\cos\chi'''\theta R''' - \quad \text{etc} \end{aligned}$$

$$\int dS.\frac{\cos q.\psi(\mu, dS)}{(\mu, dS)^2} = -\int dT.\cos\chi.\theta R$$

$$4\pi\sigma\psi_0 - \iint dt.dT.\cos\chi.\theta(dt, dT)$$

where χ indicating the mutual inclination of the element dt, dT, by the normal-direction, which is measured by the outer direction to the space s, S, which the integral by the complete surface, of which the space can be extended.

7.9. The three cases of integral.

As the same as the previous method, the division of space S in the element of prism depending is, thus the second method is necessary for the same division of space S in the element of prism. We consider

- the surface of the sphere of the radius = 1,
- and around the center μ, are descripted with the infinitesimally small elements of divised.
- Toward points, these element $d\Pi$ draw the straight line to the point μ,
- this surface of the space S are cut at the points $P', P'', P''', \cdots$;
- We denote the distances between these points $P', P'', P''', \cdots$ and μ by $R', R'', R''', \cdots$
- The straight line at μ toward all points on the peripheral elements $d\Pi$ form the pyramiid spaces, and among $P', P'', P''', \cdots$ cut the elements from the surface space S, we designate these elements with $dT', dT'', dT''', \cdots$.

Next, we assume Q' inner straight line $P'\mu$ then normal in the elements dT' extend exterior and $Q'', Q''', \cdots$ have the inclination of similar normal in the same way, drawed from the straight line toward μ. Therefore we put

$$d\Pi = \pm\frac{dT'.\cos Q'}{R'R'} = \mp\frac{dT''.\cos Q''}{R''R''} = \pm\frac{dT'''.\cos Q'''}{R'''R'''} = \cdots$$

where the sign change superior or inferior, according to that the line $\mu P'$ take interior or exterior of the space S.

[24] For example, a needle, a pin, a sting, etc.

Then, it seems clear that for all partial space of S, inside of its pyramidal space, the angle q is constant, we deduce as if it were the same as in articl 7, if we would set indefinite,

$$\int \varphi r.dr = -\theta r$$

if we assume the integral constant as arbitrary, the integral

$$\int \frac{dS.\cos\ q.\psi(\mu, dS)}{(\mu, dS)^2}$$

(I) In the case of the point μ existing in the exterior of space S :

$$\int \frac{dT.\cos\ q.\cos Q.\theta R}{R^2}$$

(II) In the case of the point μ existing in the interior of space S :

$$\theta_0. \int d\Pi.\cos q$$

(III) In the case of the point μ existing on the surface of space S :

$$\theta_0. \int d\Pi.\cos q$$

$$\cos q = \cos k.\cos v + \sin k.\sin v.\cos w$$

Integral $\int d\Pi.\cos q$ becomes

$$\int_{\frac{\pi}{2}}^{\pi} dv \int_0^{2\pi} dw \Big(\cos k.\cos v + \sin k.\sin v.\cos w \Big) \sin v$$
$$= \int_{\frac{\pi}{2}}^{\pi} 2\pi \cos k.\cos v.\sin v.dv = -2\pi \cos k \left[\frac{1}{2}\sin^2 v\right]_{\frac{\pi}{2}}^{\pi} = -\pi \cos k$$

Applied to our first integral $\iint ds.dS.\varphi(ds, dS)$ of (26), then

- (I) If the surface space s, S do not have common part, then
$$4\pi\sigma\psi_0 + \iint \frac{dt.dT.\cos\ q.\cos Q.\theta(dt, dT)}{(dt, dT)^2}$$
- (II) If the surface space s, S have common part, which is $\mathcal{T}$, then
$$4\pi\sigma\psi_0 \mp \mathcal{T}\theta_0 + \iint \frac{dt.dT.\cos\ q.\cos Q.\theta(dt, dT)}{(dt, dT)^2}$$
- (III) If the surface space s, S have plural, finite and discrete common parts, then
$$4\pi\sigma\psi_0 + \pi(\mathcal{T}' - \mathcal{T})\theta_0 + \iint \frac{dt.dT.\cos\ q.\cos Q.\theta(dt, dT)}{(dt, dT)^2}$$

7.10. Criticism of Laplace's molecular calculus of capillarity equations.

• We are almost ready to introduce two transformations of the integral $\iint ds.dS.\varphi(ds, dS)$ in the articles 8 and 9, by praising ourself, about the evolution of equations, we would like to moreover accomodate our proposition.

• Here, the function φ is used originally as the function f, for the further study built on the hyposis, on which Mr. Laplace studies, says that the force of molecular activity are more finite in the infinitesimal distance. This phrase when the liquid move adhering, how long keeps the uniformity, under everybody can observe it, the attractive activity fr, expressed by the function of distance r, and since he treats the gravity g as homogeneous, which is due to liquid mass ; this is a defect of his supposition. and denoting the liquid mass by M, whatever we can try in the experiment, and he says almost the same as nothing with respect to every part of media.

• Mfr in the infinitesimal distance is not only finite, but also even r can be decreased over the arbitray boundaries.

• Without theory and the policy to investigate that the gravity comes from the hypothesis, in the other point, the law of the function fr, as the same as the unknown in general, which we can not help making a mistake about the mathematical ≺ *character* ≻ , look like peculier : namely, as long as even the fact, standing on the most precise mathematics, can not punish himself, if so, so much as the mathematical precision, more, even without the experiments, we can get the absolute level of value ; without an experiment (or proof), none is free from the amusement by oneself in seeking after absolute truth ; if you would like to success, withdrow supposition itself. [a]

[a] Navier cites the molecular theory by Laplace and chooses consistently repulsive force in Navier's papers [25, 26] as the function depending on the distance between molecules, however, N.Bowditch [b] points out that Laplace rethinks the repulsion theory and changes it, in 1819 : $\varphi(f) = A(f) - R(f)$, where $\varphi(f)$: a function depending on the distance f between the moleculars, $A(f)$: attractive force, $R(f)$: repulsive force.

7.11. Function φr as the constant of integral $\int fr.dr$.

• Even if we suppose the function denoting by fr (or the function by Fr) of attraction, that the fact that the relation is proportional reciprocally with inverted r^2 , is proofed in the astromics , if the figure between the fluid and a vessel, in any infinitesimal particle, the gravity can also affect to the modification. r increasing in even infinitesimal, fr turns into, by itself, infinitesimal, but also more rapidly decreased rather than $\frac{1}{r^2}$.

• Hence, we can make a deduction from here as follows : even the integral $\int fr.dr$ in everywhere, it is finite, turns into infinitesimal, then that the constant of integral $\int fr.dr = -\varphi r$, is supposed to be acceptable and have $\varphi\infty = 0$, if φr this value of integral $\int_r^\infty fx.dx$ is extended.

• In any way, φr the distance denoting positive quality by r, not only infinitesimal, but also finite r ; continues to decrease with respect to the distance r, it can go beyond the arbitrary boundary, speaking by general, if there is non-obstacle, then $\varphi 0 = \infty$.

7.12. The difficulty of calcululating $\int r^2\varphi r.dr$.

• Hence, since the function φr, in everywhere, instead of the finite value of r it turns into infinitesimal, and increasing r continues to decrease, $\int r^2\varphi r.dr$ always obey to the value anywhere finite to arbitrary big extend, and moreover keep infinite, then as long as the latter, whatever we are ambitious, even if any experiments can teach us, it is only as follows : about how to make the infinitesimal integral, even by the big interval, integral is not successful.

• The very calculations by Mr. Laplace show us all these situations, in which my supposition is included ; since nature of the unknown function φr is suggestive, and using it, we can supersede it or abstain from it to many suppositional hypotheses.

• This constant of integral $\int r^2\varphi r.dr = -\psi r$ determines as we choice it, makes $\psi r = 0$, for the value of fluid with the finite distance of r, moreover, by its experiment, we can afford to get the length of circumference of the body.

• Hence, ψr for all this sort of value will be always finite (positive for minimum, negative for maximum), speaking in general, if there is non-obstacle, then for the infinitesimal value of r, we can convert to the finite value : although ought to add, we give an explanation to the phenomenon, as the decreasing distance r in infinitesimal, the value ψr itself means always as finite, as long as $\psi 0$ depends on the finite quantity.

• Besides these, $\frac{c\psi r}{r}$ is the quantity when the gravity is homogeneous, $\frac{c\psi r}{g}$ is linea, especially, $\frac{c\psi\theta}{g}$ is already-known-linea (for natural body, in this case, the function fr is useful for the force of the attractive activity), of which the magnitude may be very suspective, however, in the known case, it is an almost-approximate value, except for suppositional hypothesis.

7.13. Proof of that $\frac{\theta_0}{\psi_0}$ is linear in insensible magnitude and its avoidance.

Completely similar in integral

$$\int \psi r = -\theta r \tag{27}$$

here we suppose

- the constant as selected and $\theta r = 0$ for value arbitrary r inter the circumference of it, for this we can set how we get the way insensible θr is for any sensible value r in everywhere, even if it evaluate sensible for the insensible value.
- We assume $\frac{c\theta r}{g}$ explains the area of dual-dimensional figure, in paticular, $\frac{\theta r}{\psi r}$ is linear.
- Naturally, another $\frac{\theta_0}{\psi_0}$ is linear in insensible magnitude, which we prove as follows.

When ψr from $r = 0$ continues decrease, and certainly, such as, insensible have gone, as soon as r get sensible value, for $\psi r = \frac{1}{2}\psi_0$, [25] must be insensible : denote this value of r by ρ. We would consider the integral $\int(\psi_0 - \psi r)dr$, which we integrate it from $r = 0$ to $r = R$, it becomes from (27),

$$\int_0^R (\psi_0 - \psi r)dr = \Big[\psi_0 r + \theta r\Big]_0^R = R\psi_0 - \theta_0 + \theta R. \tag{28}$$

Cleary, this integral more greater, when it is integrated from $r = \rho$ to $r = R$, the extension becomes at any times greater than the integral $\int(\psi_0 - \psi\rho)dr$ between the same limite. The last integral becomes

$$\int_\rho^R (\psi_0 - \psi\rho)dr = (\psi_0 - \psi\rho)(R - \rho) = \frac{1}{2}\psi_0.(R - \rho) \tag{29}$$

which is generalized for this value $R(> \rho)$ from (28) and (29),

$$R\psi_0 - \theta_0 + \theta R > \frac{1}{2}\psi_0.(R - \rho)$$

If $R = \frac{\theta_0}{\psi_0}$, and moreover, if R is the sensible quantity, then

$$\theta R > \frac{1}{2}\psi_0.(R - \rho)$$

this becomes absurd value.

> Solving method : If we can not avoid this tremendus magnitude of ψ_0, by cutting only zero, θ_0 is possible to be the usually sufficient quantity and to be comparable with the dimension of body in carrying out an experiment. (If so, we get the same situation as a usual condition of experiment)

7.14. Integral (I) and (II).

Moreover, that comes from this ≺ "*indole*" function : θ ≻ with respect to the integral (I)

$$\text{integral (I)} : \iint \frac{dt.dT \cos q. \cos Q.\theta(dt, dT)}{(dt, dT)^2}$$

follows and we would like to investigate it. We begin this investigation with simplification, to be able to alternate the surface points μ, consider specially the integral (II)

$$\text{integral (II)} : \int \frac{dt. \cos q. \cos Q.\theta(\mu, dt)}{(\mu, dt)^2}$$

by all the superface : t, we consider to extend it. We denote as follows :

- Q the angle between two rectangles comming out at the point μ,
- the alternated toward the element dt,
- the alternated toward the fixed;

samely,

- q the angle between two rectangles comming out at the point dt
- the alternated toward the element μ,
- the alternated normal element toward the exterior direction
- At first, we observe, if point μ is sensible in the distance on the surface : t, all value $\theta(\mu, dt)$ is insensible : in this case, total integral (II) are insensible. Here we can get sensible value in this integral, how long we can extend the surface t in insensible distance at point μ posit, clearly enough the integral (II) by this part, all neglected, that is sensible in distance.

[25] Which we say a half-life of the radiation.

- Next, for $\frac{dt.\cos q}{(\mu,dt)^2}$, we restore by $\pm d\Pi$, and denote $d\Pi$ in the surface of the sphre with the radius $= 1$, with the center : μ, the description of element id, in which the element dt of the exterior or interior plane, direct the point μ.

Here we get the integral (II) as follows

$$\int \pm d\Pi.\cos Q.\theta(\mu, dt)$$

here it is clear that this integral the value as long it is able to be sensible , as long as all the elements $d\Pi$, at the insensible distance (μ, dt), is refered, the sensible magnitude of space in the surface filled in the sphere. We consider the following three cases.

(1) In which, the radius of curvature of the surface t is infinitesimal at the point μ.
(2) In which, the continuous curvature at the point μ which the inner distance is infinitesimal. (cf. [9]).
(3) In which, the radius : of curvature of the surface t is open at the point of μ.

7.15. Integral (II).

The integral (II) :

$$\int dv \int_0^{2\pi} dw \Big[\pm (\cos k.\cos v + \sin k.\sin v.\cos w)\theta r.\sin v\Big]$$

We denote this minimum distance with ρ (to this point G, $v = 0$ responds), $r = \frac{\rho}{\cos v}$, when $v = 0$, then $r = \rho$. We integrate from $w = 0$ to $w = 2\pi$, then

$$\pm \int 2\pi\theta r.\cos k.\cos v.\sin v.dv = \pm \int \frac{2\pi \cos k.\rho^2\theta r.dr}{r^3}$$

The integral is from $v = 0,\ r = \rho$ to the sensibly small value.

$$\pm \int \frac{2\pi \cos k.\rho^2\theta r.dr}{r^3} = \pm\pi\cos k\Big(2r^2 \int \frac{\theta r.dr}{r^3}\Big)$$

We consider generally :

$$2r^2 \int \frac{\theta r.dr}{r^3} = -\theta' r \tag{30}$$

And considering $\int \frac{\theta r.dr}{r^3} = 0$, we neglect the insensibles then the integral (II) :

$$= \pm\pi\cos k.\theta'\rho \tag{31}$$

If it seem to be doubtful, or to be right, we have the partial surface t interinsensible distance, to the point μ position for the plane, and consider this location of the sphere, and R the distance from the center of the sphere to the point μ taking as positive or negative, according to the condition that the center is in the direction toward G or in oppositional direction. Therefore we get followings :

$$\begin{cases} \cos v = \frac{\rho}{r}(1 - \frac{\rho}{2R}) + \frac{r}{2R} \\ \sin v.dv = [\frac{\rho}{r^2}(1 - \frac{\rho}{2R}) + \frac{1}{2R}]dr \end{cases}$$

from anywhere, if the mode R is the sensible quantity, we can see easily that the integral for this case, is not different with the above-mentioned in (31), sensible quantity about the value, $\pm\pi\cos k.\theta'\rho$.

- Another is the curvature of the surface t in its part, come from this, as long as the radius of curvature is insensitive, always we can assign the dual surface of a sphere, surface t in this point μ, the nearest point by tangential angle, inter this t set,
- and these radii are sensible magnitudes, clearly, then our integral inter integral fall into the related surface,
- and therefore, we could explain without sensible error, by the same formula,
- which, not only above things, but also we would suffer from the exceptions, when the surface t in the insensible distance to the point μ, would offer even the curvature of insensible radius, or aciform type, or the cusp [26].

[26] See the footnote above in the last line of §.7.

7.16. Integral (I).

Therefore it is clear that the transform come out from the integral (II) to integral (I), here insensible occure not only in this case, but also when the sensible value is produced for null point of the surface T, but also when the complex element of the surface T, for which points the integral (II) becomes sensible, the area consists of also insensible magnitude. Which are considered rightly, the integral (I) will appear, how much is able to acqire the sensible value, how long be able to keep the partial surface T or partial sensible magnitude in the insensible distance to the positive surface t.

Our integral (I) neglecting the insensible factors :

$$= -\int \pi\theta'\rho.d\tau + \int \pi\theta'\rho.d\tau'$$

Clearly this is no interest, either the parts τ and τ' or to the surface T to t is important. The value of (26) becomes

$$\iint ds.dS.\varphi(ds, dS) = 4\pi\sigma\phi 0 - \pi T\theta 0 + \pi T'\theta 0 \underbrace{-\pi\int d\tau.\theta'\rho + \pi\int d\tau'.\theta'\rho}_{integral(I)}$$

7.17. Analysis of $\iint ds.dS.\varphi(ds, dS)$.

Therefore, we can state the origin of the function θ', i.e., from (30)

$$\frac{\theta' r}{r^2} = \int \frac{2\theta x.dx}{x^3}$$

we sum the integral from $x = r$ to the arbitrary, sensible and constant value, we denote here, this by R. Clearly this integral is minus than this $\int \frac{2\theta x.dx}{x^3}$ with the interval, this is $= \frac{\theta r}{r^2} - \frac{\theta r}{R^2}$, moeover, it is the smaller minus than $\frac{\theta r}{r^2}$. Since the infinite another, it would become as follows :

$$\int \frac{2\theta x.dx}{x^3} = -\frac{\theta x}{x^2} + \int \frac{d\theta x}{x^2} = -\frac{\theta x}{x^2} - \int \frac{\psi x.dx}{x^2}$$

is, moreover,

$$\frac{\theta' r}{r^2} = \frac{\theta r}{r^2} - \frac{\theta R}{R^2} - \int \frac{\psi x.dx}{x^2}$$

taking the integral by the same interval, that is more minus than the integral $\int \frac{\psi x.dx}{x^2}$, moreover, more minus than $\frac{\psi r}{r}$; therefore the value of $\frac{\psi' r}{r^2}$ is greater than

$$\frac{\theta r}{r^2} - \frac{\theta R}{R^2} - \frac{\psi r}{r} \quad \Rightarrow \quad \theta r - \frac{r^2.\theta R}{R^2} - r\psi r$$

the interval of $\theta' r$:

$$\text{from} \quad \theta r \quad \text{to} \quad \theta r - r^2.\frac{\theta R}{R^2} - r\psi r$$

if we differentiate this expression, r decrease infinitely, then we see clearly that this quantity can evaluate to assign minus, that is supposed for example, when ψ_0 is the finite quantity. Thus we have concluded that it is due to $\theta'_0 = \theta_0$. It is clear that, in the formula, in the previous art.16, we have get the expressin :

$$\iint ds.dS.\varphi(ds, dS) = 4\pi\sigma\phi_0 - \underbrace{\pi T\theta_0 + \pi T'\theta_0} \underbrace{-\pi\int d\tau.\theta'\rho + \pi\int d\tau'.\theta'\rho}$$

is separated into

- from $\pi T\theta_0$ to $\pi T'\theta_0$
- from $\pi\int d\tau.\theta'\rho$ to $\pi\int d\tau'.\theta'\rho$

> By using these method of solution, we can cultivate the elegant mathematical sense though we must surpass to conserve the distinction of our proposition.

7.18. Solving of variation problem.

In the application of previous survey to the evolution the second term of the expression Ω in the art. 3,in the art. 6 denote by S in the art.16 σ, T, T' will be use as $s, t, 0$, if t is the total surface of the space s, in which the fluid is filled. Therefore whenever this space extensional sensible part however insensible concentration is kept, this sort of gap (crevice), the part of the second part of the expression Ω of (25) in the art. 7.3 becomes

$$= \frac{1}{2}\pi c^2(s\phi 0 - t\theta 0)$$

The exceptions are thus assumed both :

(1) Is space s contains the insensible part of the thickness, this surface offer the dual sensible part of the liquid,
- in which we denote the alternative t',
- thick space in the neughbourhood of the infinite elements : dt' by ρ,
- by accepting the expression above terminology,

$$\pi c^2 \int \theta' \rho . dt'$$

(2) We put the $\prec$ *characteristic* $f \succ$ for the force of molecular attraction and $\prec$ *characteristic* $\succ F$. The relation with the vase oughts to yield oneself to the attractive force, we denote the functions by the $\prec$ *characteristic* $\succ$ with $\phi, \psi, \theta, \theta'$ and samely with $\Phi, \Psi, \Theta, \Theta'$ applying the same relation between F and f. The third part of the expression Ω becomes generally speaking :

$$\pi c C T \Theta 0$$

(3) If in the neighbourhood of the sensible part T' of the surface T have the thick of fluid, we denote the next term, in which infinite thick of fluid by ρ, as we accept from the experiments

$$-\pi c C T \Theta' \rho . dT'$$

(4) If the surface of the vase is contiguous except for the part T, we offer T'' in the distance we denote the next term, in which by ρ indefinite distance for points in anywhere,

$$+\pi c C T \Theta' \rho . dT''$$

In static equilibrium it is due to the maximum value, this turns into

$$-gc \int zds + \frac{1}{2}c^2 s\psi_0 - \frac{1}{2}\pi c^2 t\theta_0 + \pi c C T \Theta_0$$

In an arbitrary fluid, of which the figure is yield oneself to the space s meaning invariant, of which the expression becomes as follows :

$$\int zds + \frac{\pi c\theta_0}{2g}.t - \frac{\pi C T \Theta_0}{g}.T$$

and in an equilibrium state which is due to *minimum*. Here, we denote

$$\frac{\pi c\theta_0}{2g} \equiv \alpha^2, \quad \frac{\pi C T \Theta_0}{2g} \equiv \beta^2, \quad t \equiv T + U \tag{32}$$

and denoting by W, then

$$W \equiv \int zds + (\alpha^2 - 2\beta^2)T + \alpha^2 U \tag{33}$$

7.19. Decomposition of variation of W.

The first term of the variation of W (33) is as follows :

$$ahdh + a'h'dh',$$

and T of the second term :

$$bdh + b'dh'.$$

The last term of the variation of W (33)

$$dU \equiv 0$$

Then from (33) and above three conditions, we get dW as follows :

$$dW = ahdh + a'h'dh' - (2\beta^2 - \alpha^2)(bdh + b'dh')$$

Moreover, for the volume of the integral of fluid is invariant, then

$$adh + a'dh' = 0$$

$$dW = dh\Big[a(h - h') - (2\beta^2 - \alpha^2)(b - \frac{ab'}{a'})\Big]$$

$$h - h' = (2\beta^2 - \alpha^2)(\frac{b}{a} - \frac{b'}{a'})$$

We can assume $\frac{b}{a} >> \frac{b'}{a'}$ in comparison with $\frac{b}{a}$, then

$$h - h' = (2\beta^2 - \alpha^2)\frac{b}{a}$$

We get the maximum height h :

$$h = (2\beta^2 - \alpha^2)\frac{b}{a}$$

then

$$h' = (2\beta^2 - \alpha^2)\frac{b'}{a'}, \quad h'' = (2\beta^2 - \alpha^2)\frac{b''}{a''}, \quad \cdots$$

7.20. Deduction of Gauss' formula.

Moreover, now, with theorem in art.18, we would like to determine the ≺ "*indoles*" ≻ (nature) of the figure in equilibre, these problem are changed in evolution of the general variation, expressed with W, if the motion of the figure of the space filled with a fluid occured in only infinitesimal. If when we variation calculas of the duplicated integral for case, then even the boundary as if the variable insufficiently investigated, we could approach this precise survey to a little profound.

We consider :

- the surface, which the space s
- part U, on which all the points is determined by the coordinate x, y, z,
- these three is the distance to an arbitrary horizontal plane.

It is capable to recognize z is, for example, as the indeterminated function by x, y, these secondary partial differential by convention, if the bracket is omitted, we denote it by

$$\frac{dz}{dx}.dx, \quad \frac{dz}{dy}.dy$$

In everywhere on the surface point, we condider s with respect to the rectangle surface, normale and s to the exterior direction, the angle by cosine between this normal to the axis of rectanglar coordinate x, y and z with parallel, denote them by ξ, η and ζ. Thereby it will be :

$$\xi^2 + \eta^2 + \zeta^2 = 1 \qquad (34)$$

$$\frac{dz}{dx} = -\frac{\xi}{\zeta}, \quad \frac{dz}{dy} = -\frac{\eta}{\zeta} \tag{35}$$

The boundary of surface U become linear in itself, as the same as denoted by P, and while the motion is supposed necessarily, this element dP (as the same dU as the surface) would accept always positive. The angle by cosine, that direction of the element dP easy with the axis of coordinate x, y, z, denoted by X, Y, Z : since we do not maintain truely the sense of the ambiguous direction, then, we determinate to this as follows :

- at first, we assume that the normal direction in the element dP to the surface dU tangential and these respect to innerward put in the second,
- next, normal with respect to in the surface,
- third, we put the space s exterior,
- the constitution of system, three rectangle similar to the following location, and the coordinate axis x, y, z

Thus, we can verify easy(cf. *Disquisitiones generales circa superficies curvas*), the second equations using the angle by cosine by the direction into the axis of the coordinates x, y, z are respectively

$$\eta^0 Z - \zeta^0 Y, \quad \zeta^0 X - \xi^0 Z, \quad \xi^0 Y - \zeta^0 X \quad \Rightarrow \quad \begin{vmatrix} \delta x & \delta y & \delta z \\ X & Y & Z \\ \xi^0 & \eta^0 & \zeta^0 \end{vmatrix} \tag{36}$$

if we suppose ξ^0, η^0, ζ^0 are the values of ξ, η, ζ for the points of the element dP.

7.21. Inquiry into the variation of indivisual element.

Here we would like to supplement the preliminary. We assume the surface U is the part by an arnitrary infinitesimal perturbation.

- If sufficient we consider all the perturbation, for this boundary P always invariant, at any rate, it maintains, in this vertical surface, we can induce clearly the variation of only the third coordinate z, this problem is far easy to evalute it ;
- moreover, the maximum problem in general, in the following investigating method, considering the variable boundary, in which ambiguity and difficulty combine elegantly, bring up perturbation ; how we can show, always from the start of all, three coordinates handle the variation.

We the force as we image it, and anywhere on the surface, in which the coordinates, which are x, y, z, had substituted in another, these coordinates are $x + \delta x, y + \delta y, z + \delta z$, where $\delta x, \delta y, \delta z$ are able to regard as if these were the indeterminate functions of x, y, if these values stay infinitesimal. Now we would like to inquire into the variation of singular (indivisual) element, expressed with W. We put an arbitrary point on the surface, of which the coordinates are :

$$\begin{cases} x, \quad y, \quad z, \\ x + dx, \quad y + dy, \quad z + \frac{dz}{dx}.dx + \frac{dz}{dy}.dy, \\ x + d'x, \quad y + d'y, \quad z + \frac{dz}{dx}.d'x + \frac{dz}{dy}.d'y \end{cases} \Rightarrow \begin{vmatrix} \delta x & \delta y & \delta z \\ dx & dy & \frac{dz}{dx}.dx + \frac{dz}{dy}.dy \\ d'x & d'y & \frac{dz}{dx}.d'x + \frac{dz}{dy}.d'y \end{vmatrix}$$

If we may suppose $dx.d'y - dy.d'x > 0$, the duplex area of this triangle is made by our principle as follows :

$$(dx.d'y - dy.d'x)\sqrt{\left[1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right]}$$

First point :

$$x + \delta x, \quad y + \delta y, \quad z + \delta z$$

Second point :

$$\begin{cases} x + dx + \delta x + \frac{d\delta x}{dx}.dx + \frac{d\delta x}{dy}.dy, \\ y + dy + \delta y + \frac{d\delta y}{dx}.dx + \frac{d\delta y}{dy}.dy, \\ z + \frac{dz}{dx}.dx + \frac{dz}{dy}.dy + \delta z + \frac{d\delta z}{dx}.dx + \frac{d\delta z}{dy}.dy \end{cases} \Rightarrow \begin{cases} x + \delta x + (1 + \frac{d\delta x}{dx}).dx + \frac{d\delta x}{dy}.dy, \\ y + \delta y + \frac{d\delta y}{dx}.dx + (1 + \frac{d\delta y}{dy}).dy, \\ z + \delta z + (\frac{dz}{dx} + \frac{d\delta z}{dx}).dx + (\frac{dz}{dy} + \frac{d\delta z}{dy}).dy \end{cases}.$$

Third point :

$$\begin{cases} x + d'x + \delta x + \frac{d\delta x}{dx}.d'x + \frac{d\delta x}{dy}.d'y, \\ y + d'y + \delta y + \frac{d\delta y}{dx}.d'x + \frac{d\delta y}{dy}.d'y, \\ z + \frac{dz}{dx}.d'x + \frac{dz}{dy}.d'y + \delta z + \frac{d\delta z}{dx}.d'x + \frac{d\delta z}{dy}.d'y \end{cases} \Rightarrow \begin{cases} x + \delta x + (1 + \frac{d\delta x}{dx}).d'x + \frac{d\delta x}{dy}.d'y, \\ y + \delta y + \frac{d\delta y}{dx}.d'x + (1 + \frac{d\delta y}{dy}).d'y, \\ z + \delta z + (\frac{dz}{dx} + \frac{d\delta z}{dx}).d'x + (\frac{dz}{dy} + \frac{d\delta z}{dy}).d'y \end{cases}$$

$$\begin{cases} x + \delta x, \quad y + \delta y, \quad z + \delta z, \\ x + \delta x + (1 + \frac{d\delta x}{dx}).dx + \frac{d\delta x}{dy}.dy, \quad y + \delta y + \frac{d\delta y}{dx}.dx + (1 + \frac{d\delta y}{dy}).dy, \quad z + \delta z + (\frac{dz}{dx} + \frac{d\delta z}{dx}).dx + (\frac{dz}{dy} + \frac{d\delta z}{dy}).dy \\ x + \delta x + (1 + \frac{d\delta x}{dx}).d'x + \frac{d\delta x}{dy}.d'y, \quad y + \delta y + \frac{d\delta y}{dx}.d'x + (1 + \frac{d\delta y}{dy}).d'y, \quad z + \delta z + (\frac{dz}{dx} + \frac{d\delta z}{dx}).d'x + (\frac{dz}{dy} + \frac{d\delta z}{dy}).d'y \end{cases}$$

$$\Rightarrow \begin{vmatrix} \delta x & \delta y & \delta z \\ (1 + \frac{d\delta x}{dx}).dx + \frac{d\delta x}{dy}.dy & \frac{d\delta y}{dx}.dx + (1 + \frac{d\delta y}{dy}).dy & (\frac{dz}{dx} + \frac{d\delta z}{dx}).dx + (\frac{dz}{dy} + \frac{d\delta z}{dy}).dy \\ (1 + \frac{d\delta x}{dx}).d'x + \frac{d\delta x}{dy}.d'y & \frac{d\delta y}{dx}.d'x + (1 + \frac{d\delta y}{dy}).d'y & (\frac{dz}{dx} + \frac{d\delta z}{dx}).d'x + (\frac{dz}{dy} + \frac{d\delta z}{dy}).d'y \end{vmatrix}$$

The duplex trianglar area consisted of these points by our principle is, for brevity, denoting the sum by N, then

$$(dx.d'y - dy.d'x)\sqrt{N},$$

where,

$$N = \underbrace{\Big[\Big(1 + \frac{d\delta x}{dx}\Big)\Big(1 + \frac{d\delta y}{dy}\Big) - \frac{d\delta x}{dy}.\frac{d\delta y}{dx}\Big]}_{\text{first point}}{}^2 + \underbrace{\Big[\Big(1 + \frac{d\delta x}{dx}\Big)\Big(\frac{dz}{dy} + \frac{d\delta z}{dy}\Big) - \frac{d\delta x}{dy}\Big(\frac{dz}{dx} + \frac{d\delta z}{dx}\Big)\Big]}_{\text{second point}}{}^2$$

$$+ \underbrace{\Big[\Big(1 + \frac{d\delta y}{dy}\Big)\Big(\frac{dz}{dx} + \frac{d\delta z}{dx}\Big) - \frac{d\delta y}{dx}\Big(\frac{dz}{dy} + \frac{d\delta z}{dy}\Big)\Big]}_{\text{third point}}{}^2$$

$$= C^2 + \Big[\Big(1 + \frac{d\delta x}{dx}\Big)D - \frac{d\delta x}{dy}E\Big]^2 + \Big[\Big(1 + \frac{d\delta y}{dy}\Big)E - \frac{d\delta y}{dx}D\Big]^2$$

$$= C^2 + \Big[\Big(1 + \frac{d\delta x}{dx}\Big)^2 + \Big(\frac{d\delta y}{dx}\Big)^2\Big]D^2 + \Big[\Big(\frac{d\delta x}{dy}\Big)^2 + \Big(1 + \frac{d\delta y}{dy}\Big)^2\Big]E^2 - 2\Big[\Big(1 + \frac{d\delta x}{dx}\Big)\frac{d\delta x}{dy} + \Big(1 + \frac{d\delta y}{dy}\Big)\frac{d\delta}{d}$$

$$= (C_1 + C_2)^2 + \Big[D_1^2 + D_2^2\Big]D^2 + \Big[E_2^2 + E_1^2\Big]E^2 - 2\Big[D_1E_2 + E_1D_2\Big]DE,$$

where,

$$C = \Big(1 + \frac{d\delta x}{dx}\Big)\Big(1 + \frac{d\delta y}{dy}\Big) - \frac{d\delta x}{dy}.\frac{d\delta y}{dx} = 1 + \frac{d\delta x}{dx} + \frac{d\delta y}{dy}, \quad D = \frac{dz}{dy} + \frac{d\delta z}{dy}, \quad E = \frac{dz}{dx} + \frac{d\delta z}{dx},$$

and $C_1, C_2, D_1, D_2, E_1, E_2$ are its conjugate part. [27] And for brevity, denoting the sum by L, then

$$\sqrt{N} = \Big(\Big[1 + (\frac{dz}{dx})^2 + (\frac{dz}{dy})^2\Big].\Big[1 + \frac{L}{1 + (\frac{dz}{dx})^2 + (\frac{dz}{dy})^2}\Big]\Big)^{\frac{1}{2}}$$

where,

$$\begin{aligned} L &= \frac{d\delta x}{dx}\Big[1 + \Big(\frac{dz}{dy}\Big)^2\Big] - \frac{d\delta x}{dy}\frac{dz}{dx}\frac{dz}{dy} - \frac{d\delta y}{dx}\frac{dz}{dx}\frac{dz}{dy} + \frac{d\delta y}{dy}\Big[1 + \Big(\frac{dz}{dx}\Big)^2\Big] + \frac{d\delta z}{dx}\frac{dz}{dx} + \frac{d\delta z}{dy}\frac{dz}{dy} \\ &= \frac{d\delta x}{dx}\Big[1 + \Big(\frac{dz}{dy}\Big)^2\Big] - \frac{dz}{dx}\frac{dz}{dy}\Big(\frac{d\delta x}{dy} + \frac{d\delta y}{dx}\Big) + \frac{d\delta y}{dy}\Big[1 + \Big(\frac{dz}{dx}\Big)^2\Big] + \frac{d\delta z}{dx}\frac{dz}{dx} + \frac{d\delta z}{dy}\frac{dz}{dy} \end{aligned} \tag{38}$$

From (34) and (35), i.e.,

$$\xi^2 + \eta^2 + \zeta^2 = 1, \quad \frac{dz}{dx} = -\frac{\xi}{\zeta}, \quad \frac{dz}{dy} = -\frac{\eta}{\zeta}, \quad 1 + \Big(\frac{dz}{dx}\Big)^2 + \Big(\frac{dz}{dy}\Big)^2 = \frac{1}{\zeta^2},$$

---

[27] In *Disquisitiones generales circa superficies curvas*, Gauss deduces the following concluding equation (cf. (19), (20)) :

$$EG - F^2 = E(\frac{dr}{dq})^2 - 2F.\frac{dr}{dp}.\frac{dr}{dq} + G(\frac{dr}{dp})^2, \qquad (E.\frac{dr}{dq} - F.\frac{dr}{dp}).\frac{d\varphi}{dq} = (F.\frac{dr}{dq} - G.\frac{dr}{dp}).\frac{d\varphi}{dp}$$

We see N of (37) resembles (19).

the ratio of the first triangle to the second and plus 1 becomes,

$$1+\frac{L}{1+(\frac{dz}{dx})^2+(\frac{dz}{dy})^2}=1+\zeta^2 L$$

Moreover, this is independent of the figure of triangle dU, then, it turns out,

$$\delta dU=\frac{LdU}{1+(\frac{dz}{dx})^2+(\frac{dz}{dy})^2}=\zeta^2 LdU$$

Expanding L with (38), then

$$\delta dU=dU\Big[\frac{d\delta x}{dx}\big(\eta^2+\zeta^2\big)-\big(\frac{d\delta x}{dy}+\frac{d\delta y}{dx}\big)\xi\eta+\frac{d\delta y}{dy}\big(\xi^2+\zeta^2\big)-\frac{d\delta z}{dx}\xi\zeta-\frac{d\delta z}{dy}\eta\zeta\Big]$$

7.22. Decomposition of U into A and B.

All variation of the surface U is obtained by the following two integrals

$$\int dU\Big[\big(\eta^2+\zeta^2\big)\frac{d\delta x}{dx}-\xi\eta\big(\frac{d\delta y}{dx}\big)-\xi\zeta\frac{d\delta z}{dx}\Big]\equiv A, \tag{39}$$

$$\int dU\Big[-\xi\eta\frac{d\delta x}{dy}+\big(\xi^2+\zeta^2\big)\frac{d\delta y}{dy}-\eta\zeta\frac{d\delta z}{dy}\Big]\equiv B \tag{40}$$

separately treated. We consider as follows :

- we take the plane, rectangle to the coordinate axis y, and such as, the value determinated by itself, suitable it, it is between peripheral, the last value, which y has in the surface U.
- For this plane, on the peripheral P, we cut in two part, or four, or six, etc., the points, of which the first coordinate will be followed by $x^0, x', x'', \cdots$;
- as if the other quantities, we put suitablly the indicies for these points ;
- by the same way, we cut the surface with other plane, this infinite neighbourhood and parallel, which encounters with the second coordinate at the point of $y+dy$;
- between these planes, we could get the elements of peripheral $dP^0, dP', dP'', \cdots$,

then we could see easy having followings :

$$dy=-Y^0dP^0=+Y'dP'=-Y''dP''=+Y'''dP''' \text{ etc.} \tag{41}$$

If, in addition to, we consider the infinitely many planes, rectangle to the coordinate axis x, of which the element dx between x^0 and x', or between x'' and x''', or etc., it corresponds to the element :

$$dU=\frac{dx.dy}{\zeta}, \tag{42}$$

therefore, from here, it is clear for a part of the integral : A, that corresponds to the part of the surface depending on between the interval : $y,\ y+dy$, to have by the following integral, i.e., substituting the right hand-side of (42) into A of (39), then

$$A=dy\int dx\Big(\frac{\eta^2+\zeta^2}{\zeta}.\frac{d\delta x}{dx}-\frac{\xi\eta}{\zeta}.\frac{d\delta y}{dx}-\xi d\delta z\Big)$$

extending from $x=x^0$ to $x=x'$, next $x=x''$ to $x=x'''$ etc. More, infinite this integral is as follow

$$A=\Big(\frac{\eta^2+\zeta^2}{\eta}\delta x-\frac{\xi\zeta}{\zeta}\delta y-\xi\delta z\Big)dy-dy\int\Big(\delta x\frac{d\frac{\eta^2+\zeta^2}{\zeta}}{dx}-\delta y\frac{d\frac{\xi\eta}{\zeta}}{dx}-\delta z\frac{d\xi}{dx}\Big)dx$$

Here, we construct A using (41), then

$$\begin{aligned}
&\Big(\frac{\eta^{02}+\zeta^{02}}{\zeta^0}\delta x^0-\frac{\xi^0\eta^0}{\zeta^0}\delta y^0-\xi^0\delta z^0\Big)Y^0dP^0\\
+\ &\Big(\frac{\eta'^2+\zeta'^2}{\zeta'}\delta x'-\frac{\xi'\eta'}{\zeta'}\delta y'-\xi'\delta z'\Big)Y'dP'\\
+\ &\Big(\frac{\eta''^2+\zeta''^2}{\zeta''}\delta x''-\frac{\xi''\eta''}{\zeta''}\delta y''-\xi''\delta z''\Big)Y''dP''\\
+\ &\text{etc.}\\
-\ &\int\zeta dU\Big(\delta x\frac{d\frac{\eta^2+\zeta^2}{\zeta}}{dx}-\delta y\frac{d\frac{\xi\eta}{\zeta}}{dx}-\delta z\frac{d\xi}{dx}\Big)
\end{aligned}$$

or in sum,

$$\sum\left(\frac{\eta^2+\zeta^2}{\zeta}\delta x-\frac{\xi\eta}{\zeta}\delta y-\xi\delta z\right)YdP-\int\zeta dU\left(\delta x\frac{\frac{\eta^2+\zeta^2}{\zeta}}{dx}-\delta y\frac{d\frac{\xi\eta}{\zeta}}{dx}-\delta z\frac{d\xi}{dx}\right)$$

This total quantity A is expressed by

$$A=\int\left(\frac{\eta^2+\zeta^2}{\zeta}\delta x-\frac{\xi\eta}{\zeta}\delta y-\xi\delta z\right)YdP-\int\zeta dU\left(\delta x\frac{\frac{\eta^2+\zeta^2}{\zeta}}{dx}-\delta y\frac{d\frac{\xi\eta}{\zeta}}{dx}-\delta z\frac{d\xi}{dx}\right) \quad (43)$$

where, the first integral is extended to all the circumference of P, and the second is extended to all the surface of U.

7.23.. Reduction of δU with Q and V via A and B.

By calculation from (39) as the same as (43), we get B similarly and immediately

$$B=\int\left(\frac{\xi\eta}{\zeta}\delta x-\frac{\xi^2+\zeta^2}{\zeta}\delta y-\eta\delta z\right)XdP+\int\zeta dU\left(\delta x\frac{\frac{\xi\eta}{\zeta}}{dy}-\delta y\frac{d\frac{\xi^2+\zeta^2}{\zeta}}{dy}+\delta z\frac{d\eta}{dy}\right) \quad (44)$$

Here we determine for all the circumference P, we get ζQ from the first terms of both (43) and (44),

$$\left(\frac{\eta^2+\zeta^2}{\zeta}\delta x-\frac{\xi\eta}{\zeta}\delta y-\xi\delta z\right)Y+\left(\frac{\xi\eta}{\zeta}\delta x-\frac{\xi^2+\zeta^2}{\zeta}\delta y-\eta\delta z\right)X\equiv Q$$

$$\left[X\xi\eta+Y\left(\eta^2+\zeta^2\right)\right]\delta x-\left[X(\xi^2+\zeta^2)+Y\xi\eta\right]\delta y+(X\eta\zeta-Y\xi\zeta)\delta z=\zeta Q$$

Moreover, we determine for every point of the surface U, we get V from the second terms of both (43) and (44),

$$\left(\frac{d\frac{\xi\eta}{\zeta}}{dy}-\frac{d\frac{\eta^2+\zeta^2}{\zeta}}{dx}\right)\zeta\delta x+\left(\frac{d\frac{\xi\eta}{\zeta}}{dx}-\frac{d\frac{\xi^2+\zeta^2}{\zeta}}{dy}\right)\zeta\delta y+\left(\frac{d\xi}{dx}+\frac{d\eta}{dy}\right)\zeta\delta z=V \quad (45)$$

That is, we can put

$$\delta U=\int QdP+\int VdU \quad (46)$$

The first integral is by all the circumference P, and the second is by all surface U.

7.24. Reduction of $\frac{d\xi}{dx}+\frac{d\eta}{dy}$ from V.

Formula for Q and V notably contradict $X\xi+Y\eta+Z\zeta=0$, Q has always the symmetric form as follows :

$$Q = (Y\zeta-Z\eta)\delta x+(Z\xi-X\zeta)\delta y+(X\eta-Y\xi)\delta z$$

$$\Rightarrow \text{ outer product of } \begin{vmatrix}\delta x & \delta y & \delta z\\ X & Y & Z\\ \xi & \eta & \zeta\end{vmatrix}$$

When we see the form of V, we can reduce from the formulae (35)

$$\frac{dz}{dx}=-\frac{\xi}{\zeta},\quad \frac{dz}{dy}=-\frac{\eta}{\zeta}$$

the following as

$$\frac{d\frac{\xi}{\zeta}}{dy}=\frac{d\frac{\eta}{\zeta}}{dx} \quad (47)$$

therefore,

$$\frac{d\frac{\xi\eta}{\zeta}}{dy} = \frac{\xi}{\zeta}.\frac{d\eta}{dy} + \eta\frac{d\frac{\xi}{\zeta}}{dy} = \frac{\xi}{\zeta}.\frac{d\eta}{dy} + \eta\frac{d\frac{\eta}{\zeta}}{dx}$$

Moreover, for $\xi^2+\eta^2+\zeta^2=1$, we can deduce

$$\xi\frac{d\xi}{dx} + \eta\frac{d\eta}{dx} + \zeta\frac{d\zeta}{dx} = 0 \tag{48}$$

by dividing the both side of hand of (48) with ζ,

$$\frac{\xi}{\zeta}\frac{d\xi}{dx} = -\Big(\frac{\eta}{\zeta}\frac{d\eta}{dx} + \frac{d\zeta}{dx}\Big) \tag{49}$$

and therefore by (49)

$$\frac{d\frac{\eta^2+\zeta^2}{\zeta}}{dx} = \eta\frac{d\frac{\eta}{\zeta}}{dx} + \Big(\frac{\eta}{\zeta}.\frac{d\eta}{dx} + \frac{d\zeta}{dx}\Big) = \eta\frac{d\frac{\eta}{\zeta}}{dx} - \frac{\xi}{\zeta}.\frac{d\xi}{dx} \tag{50}$$

We may replace the coefficient of $\zeta\delta x$ in V of (45), using (47) and (50),

$$\begin{aligned}
&\frac{d\frac{\xi\eta}{\zeta}}{dy} - \frac{d\frac{\eta^2+\zeta^2}{\zeta}}{dx} \\
&= \frac{d\frac{\xi\eta}{\zeta}}{dy} - \eta\frac{d\frac{\eta}{\zeta}}{dx} + \frac{\xi}{\zeta}.\frac{d\xi}{dx} \quad \text{(, from (50),)} \\
&= \Big(\frac{\xi}{\zeta}\frac{d\eta}{dy} + \eta\frac{d\frac{\xi}{\zeta}}{dy}\Big) - \eta\frac{d\frac{\xi}{\zeta}}{dy} + \frac{\xi}{\zeta}.\frac{d\xi}{dx} \quad \text{(, from (47),)} \\
&= \frac{\xi}{\zeta}\Big(\frac{d\xi}{dx} + \frac{d\eta}{dy}\Big)
\end{aligned}$$

Samely for $\zeta\delta y$

$$\frac{d\frac{\xi\eta}{\zeta}}{dx} - \frac{d\frac{\xi^2+\zeta^2}{\zeta}}{dy} = \frac{\eta}{\zeta}\Big(\frac{d\xi}{dx} + \frac{d\eta}{dy}\Big)$$

Then V of (45) is reduced as follows :

$$V = (\xi\delta x + \eta\delta y + \zeta\delta z)\Big(\frac{d\xi}{dx} + \frac{d\eta}{dy}\Big)$$

7.25. Geometric meaning of $\frac{d\xi}{dx} + \frac{d\eta}{dy}$ in V.

Before going forward, we must illustrate conveniently the important geometrical expression. Here we restrict the various direction, we would like to present the following its intuitionally facile method, which we introduced in *Disquisitiones generales circa superficies curvas*. We consider the following layout of structure.

- We put the sphere, of which the radius = 1 at the center of an arbitrary surface,
- we denote the axis of the coordinates x, y and z by the points (1), (2) and (3),
- next, taking exterior domain denoted by s, we number a point denoting by the point (4) toward the normal direction on surface ;
- moreover, at an arbitrary point on surface, drawing various rectangle direction toward point of itself, we denote by the point (5),
- for the variation of itself, we suppose that the quantity $\sqrt{\delta x^2+\delta y^2+\delta z^2}$ is always positive, and we denote with δe for brevity, then

$$\begin{cases} \delta x = \delta e.\cos(1,5) \\ \delta y = \delta e.\cos(2,5) \\ \delta z = \delta e.\cos(3,5) \end{cases}$$

Here, we would like to express the every point on the surface. In this boundary, if we call the periphery P, we can consider the two directions.

- At first, we denote the corresponding point to dP by the point (6),

- next, we draw the rectangle direction to the surface, which is the inner normally-directed tangential to the surface, then we denote the point by (7),
- by the hypothesis, our points (6), (7), (4) look toward the same direction , [28]
- using above-mentionhed (1), (2) and (3) then (4, 6), (4, 7) and (6, 4) make a cube, [29] when we consider the angles as the rectangles.

Thus, the equations (36) in the above-mentioned (§20) transform into

$$\begin{cases} \eta Z - \zeta Y = \cos(1,7) \\ \zeta X - \xi Z = \cos(2,7) \\ \xi Y - \eta X = \cos(3,7) \end{cases}$$

Namely $\cos(1,7)$, $\cos(2,7)$, $\cos(3,7)$ are determined by the following determinant using its cofactor :

$$\begin{vmatrix} \cos(1,7) & \cos(2,7) & \cos(3,7) \\ X & Y & Z \\ \xi & \eta & \zeta \end{vmatrix}$$

In the previous article, these forms are as follows :

$$Q = -\delta e.\cos(5,7), \quad V = \delta e.\cos(4,5).\Big(\frac{d\xi}{dx} + \frac{d\eta}{dy}\Big) \tag{51}$$

Here

- Q is the translation of this point in the periphery P, this tangential surface U, normal in the domain, taking as positive to the opposite direction ;
- the factor V is, as $\cos(4,5)$ cleary indicates, the translation of this point on the surface U, taking as positive in the domain of the exterior space s.

We may explain by replacing $\frac{d\xi}{dx} + \frac{d\eta}{dy}$ in V of (51), from the point of view in geometric meaning. In such case, it turns namely as follows :
From (35)

$$\xi = -\zeta.\frac{dz}{dx}, \quad \eta = -\zeta.\frac{dz}{dy} \tag{52}$$

$$\rightarrow \quad \xi^2 + \eta^2 + \zeta^2 = \zeta^2 + \zeta^2\Big(\big(\frac{dz}{dx}\big)^2 + \big(\frac{dz}{dy}\big)^2\Big)$$

Then

$$\frac{1}{\zeta^2} = 1 + \big(\frac{dz}{dx}\big)^2 + \big(\frac{dz}{dy}\big)^2 \tag{53}$$

Taking derivative in both side of hand of (53)

$$-2\zeta^{-3} = 2\frac{dz}{dx}.\frac{d\frac{dz}{dx}}{d\zeta} + 2\frac{dz}{dy}.\frac{d\frac{dz}{dy}}{d\zeta}$$

$$1 = -\zeta\frac{dz}{dx}.\zeta^2\frac{d\frac{dz}{dx}}{d\zeta} - \zeta\frac{dz}{dy}.\zeta^2\frac{d\frac{dz}{dy}}{d\zeta} \tag{54}$$

and finally we get the following after replacing (54) with ξ and η from (52)

$$d\zeta = \xi\zeta^2 d\frac{dz}{dx} + \eta\zeta^2 d\frac{dz}{dy} \tag{55}$$

[28]This image is considered that there are three directions emitting from a common point and making a certain angle with two directions (i.e. points.)

[29](4, 6), (4, 7) and (6, 4) make a plane consisting of a cube respectively.

[30] Using (52) and (55),

$$
\begin{aligned}
\frac{d\xi}{dx} &= -\zeta\frac{d^2z}{dx^2} - \frac{dz}{dx}.\frac{d\zeta}{dx} \\
&= -\zeta\frac{d^2z}{dx^2} \underbrace{-\zeta\frac{dz}{dx}}_{=\xi}\xi\zeta\frac{d^2z}{dx^2} + \xi\eta\zeta\frac{d^2z}{dx.dy} \\
&= -\zeta(1-\xi^2)\frac{d^2z}{dx^2} + \xi\eta\zeta\frac{d^2z}{dx.dy} \\
&= -\zeta(\eta^2+\zeta^2)\frac{d^2z}{dx^2} + \xi\eta\zeta\frac{d^2z}{dx.dy}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\eta}{dy} &= -\zeta\frac{d^2z}{dy^2} + \eta^2\zeta\frac{d^2z}{dy^2} + \xi\eta\zeta\frac{d^2z}{dx.dy} \\
&= -\zeta(1-\eta^2)\frac{d^2z}{dy^2} + \xi\eta\zeta\frac{d^2z}{dx.dy} \\
&= -\zeta(\xi^2+\zeta^2)\frac{d^2z}{dy^2} + \xi\eta\zeta\frac{d^2z}{dx.dy}
\end{aligned}
$$

Therefore, again from (52)

$$
\begin{aligned}
\frac{d\xi}{dx} + \frac{d\eta}{dy} &= -\zeta^3\Big[\frac{d^2z}{dx^2}\Big\{1+\Big(\frac{dz}{dy}\Big)^2\Big\} - \frac{2d^2z}{dx.dy}.\frac{dz}{dx}.\frac{dz}{dy} + \frac{d^2z}{dy^2}\Big\{1+\Big(\frac{dz}{dx}\Big)^2\Big\}\Big], \\
\text{where,}\quad \zeta^3 &= \Big[1+\big(\frac{dz}{dx}\big)^2+\big(\frac{dz}{dy}\big)^2\Big]^{-\frac{3}{2}}
\end{aligned}
$$

This is equal to (19) in Gauss [9]. [31] This value turns into a constant such as [32]

$$
\frac{d\xi}{dx} + \frac{d\eta}{dy} = \frac{1}{R} + \frac{1}{R'}, \tag{56}
$$

where R and R' are the radii of curvature respectively. [33]

---

[30]The above expressions are to be used by ∂, that is

$$
\partial\zeta = \xi\zeta^2\partial\frac{\partial z}{\partial x} + \eta\zeta^2\partial\frac{\partial z}{\partial y}
$$

[31]Kobayashi[15], p.138 (3.9), *the first fundamental form* :

$$
\begin{aligned}
I_\alpha &= E_\alpha du_\alpha du_\alpha + 2F_\alpha du_\alpha dv_\alpha + G_\alpha dv_\alpha dv_\alpha \\
&= (du_\alpha,\ dv_\alpha)\begin{bmatrix} E_\alpha & F_\alpha \\ F_\alpha & G_\alpha \end{bmatrix}\begin{bmatrix} du_\alpha \\ dv_\alpha \end{bmatrix}
\end{aligned}
$$

where,

$$
E_\alpha = \frac{\partial \mathrm{p}}{\partial u_\alpha}.\frac{\partial \mathrm{p}}{\partial u_\alpha},\quad F_\alpha = \frac{\partial \mathrm{p}}{\partial u_\alpha}.\frac{\partial \mathrm{p}}{\partial v_\alpha},\quad G_\alpha = \frac{\partial \mathrm{p}}{\partial v_\alpha}.\frac{\partial \mathrm{p}}{\partial v_\alpha}
$$

[32]cf. Laplace, IV, p.826 [9853], the equation (3) :

$$
\frac{1}{R} + \frac{1}{R'} = \frac{(1+q^2).\frac{dp}{dx} - pq.\left(\frac{dp}{dy}+\frac{dq}{dx}\right) + (1+p^2).\frac{dq}{dy}}{(1+p^2+q^2)^{\frac{3}{2}}}
$$

[33]cf. Poisson [29], p.105.

7.26. Reduction of δU.

From (46), (51) and (56)

$$(I)\quad \delta U = \int Q dP + \int V dU = -\int \underbrace{\delta e.\cos(5,7)}_{-Q\ of\ (51)} .dP + \int \underbrace{\delta e.\cos(4,5).(\frac{1}{R} + \frac{1}{R'})}_{V\ of\ (51)} dU$$

Now, we consider Q of (51),

$$\int \delta e.\cos(5,7)dP' = \int \delta e.\cos(5,7)dP^{(3)} + \int \delta e.\cos(5,7)dP^{(4)} \tag{57}$$

$$\int \delta e.\cos(5,7)dP^{(2)} = \int \delta e.\cos(5,7)dP^{(3)} + \int \delta e.\cos(5,7)dP^{(5)} \tag{58}$$

We add both hand-sides of two equations (57) and (58) above, then

$$\int \delta e.\cos(5,7)dP' + \int \delta e.\cos(5,7)dP^{(2)} = \int \delta e.\cos(5,7)dP$$

For $\delta U = \delta U' + \delta U''$, the variational values of $\delta U'$, $\delta U''$ are fit by substitution. Thus, we can see the truth of the formula (I).

7.27. Setting the positions.

We consider in art. 21 that $\xi\delta x + \eta\delta y + \zeta\delta z = \delta e.\cos(4,5)$ then

$$(II)\quad \delta s = \int dU.\delta e.\cos(4,5)$$

$$(III)\quad \delta \int z ds = \int z dU.\delta e.\cos(4,5)$$

We denote with point (8), the responding direction, surfacial variation T transpositioning element, we get $dP.\delta e.\cos(5,8)$ from dP, namely (IV) :

$$(IV)\quad \delta T = \int dP.\delta e.\cos(5,8),$$

where, the sign of factor $\cos(5,8)$ is decided according to the conditions of whether increment or decrement.

integral with the total linear P extend, then $(5,7) = \cos(5,8).\cos(7,8)$: the arc $(7,8)$ moreover the measure of angle between two planes of the surface spaces s, S tangential on its intersection P, and the interplane domain, which include null space. Here we denote the angle with i, and with $2\pi - i$, we denote interplane domain ofcontinuing space of s. We introduce namely $(7,8) \equiv i$ as the boundary angle, then we formulate (V) as follows :

$$(V)\quad \cos(5,7) = \cos(5,8).\cos i$$

7.28. Variation δW and reduction from the first fundamental theorem.

E the combination of formulae I, $\cdots$, IV, we get variational expression W.

$$\delta W = \underbrace{\int dU.\delta e.\cos(4,5)}_{(II)\ \delta s} .\left[z + \alpha^2\left(\frac{1}{R} + \frac{1}{R'}\right)\right] - \underbrace{\int dP.\delta e.\cos(5,8)}_{(IV)\ \delta T} .(\alpha^2 \cos\ i - \alpha^2 + 2\beta^2)$$

where,

$$z + \alpha^2\left(\frac{1}{R} + \frac{1}{R'}\right) = \text{Const.}$$

The equation is constituted by ≺ *the first fundamental theorem* ≻, in the theory of fluid equilibrium, in which Mr. Laplace missed, however, it would come to be different if he had used our method.

If we set Const = 0, then

$$z = -\alpha^2\Big(\frac{1}{R} + \frac{1}{R'}\Big).$$

where, z is the height of capillary action. And moreover, the following corollaries as follows :

Corollaries :

(1) If free surface U is not classified, in any point in a sction, the surface must be concavo-convex, (i.e. concave curvature is greater than convex curvature,) in addition, convexing the maximum radius is equivalent to concaving with the maximum radius.

(2) For upper normal plane to surface, it becomes concavo-concave, (i.e. biconcave, which is concave in both sides,) or if there is in anywhere, convexo-concave, (i.e. convex curvature is greater than concave curvature,) concave curvature will be convex.

(3) It becomes convexo-convex, (i.e. biconvex, which is convex in both sides,) or if there is in anywhere, concavo-convex, convex curvature will be concave.

(4) Free surface U can not have partial finite plane if not horizontal and coincident with normal plane.

7.29. Reduction from the second fundamental theorem.

$$\delta W = -\int dP.\delta e.\cos(5,8).(\alpha^2\cos i - \alpha^2 + 2\beta^2) = \alpha^2\int dP.\delta e.\cos(5,8).\Big(1 - 2\big(\frac{\beta}{\alpha}\big)^2 - \cos\ i\Big)$$

Here, we introduce A such that

$$\cos A = 1 - 2\sin^2(\frac{A}{2}) = 1 - 2\frac{\beta^2}{\alpha^2} \quad \text{if} \quad \sin\frac{A}{2} = \frac{\beta}{\alpha}, \tag{59}$$

then

$$\delta W = \alpha^2\int dP.\delta e.\cos(5,8).(\cos A - \cos i)$$

integration is extended along the total line P. Remember that the factor $\cos(5,8)$ is equivalent with $\sin(5,6)$, [34] and the sign becomes plus or minus, according to fluid in motion in the neighbourhood of element dP or moreover, it reachs to the end point of P, or it comes to disappear. Here, we conclude that as follows :

- in state of equilibrium, it becomes always $i = A$.
- If in every part of the line P, it becomes $i < A$, then initially generated momentum in this part keeps invariable in the line P, and W show negative variation.
- If in a part of the line P, it becoms $i > A$, then both cases of minimum condition and equilibrium confront.

This is ≺ *the second fundamental theorem* ≻, which Mr.Laplace has investigated almost without proof in the meaning of the principle of molecule.

7.30. In case of the vase having the figure of cusp or aciform.

- The theorem above of arrangement which lacks in singular case, we can not pass over it.
- The surface of the vase near the ultimate limit P, such that in this limit point, there exists the *only* plane contact with the surface of vase.
- If the continuous curvature in this point P the singular line interrupted, it is considered easily that not only the cusp, but also the aciform [35] of line P sifts, we do not change our conclusions ;

[34] i.e. $\cos(5,8) = \sin(5,6)$, where the point (8) is the point of rectangle, the points (6), (8) and (5) make a straight line in the direction from left to right.

[35] For example, a needle, a pin, a sting, etc. See the footnote above in the last line of § 7.7.

$$\alpha^2 dP.\delta e. \sin(5,6).(\cos A - \cos i)$$

$$-\alpha^2 dP.\delta e. \sin(5,6).(\cos A + \cos k)$$

$$\begin{cases} i = A, & i > A \\ k = 2\pi - A, & k > 2\pi - A \end{cases}$$

In the state of equilibrium, therefore, it can not become $i + k < 2\pi$, if, that is equivalent to the following : *in the state of equilibrium, the limit of free surface of fluid can not become up to the finite extension, in the aciform, concave surface of vase.* To the contrary, the quantities by this limit coincident with aciform convex, this is required and sufficient for equilibrium, where, a is the inclination.

- When the angle lies between fluid plane and tangent vase as follows :

$$\begin{cases} \text{between } A \text{ and } A + a, & \text{exterior fluid,} \\ \text{between } 2\pi - A \text{ and } 2\pi - A + a, & \text{interior measureable fluid} \end{cases}$$

- When the angle lies between two surface planes of vase from both side to aciform tangent in this point indefined denoted with $2\pi - \alpha$, to what extent we can measure this angle of domain of vase.

7.31. Relations of quantities of attractions between fluid and vase in respect to the angle A.

The constant α^2 and β^2, which ratio of the angle A determinated depending on the function f and F, and in a sense, we can consider as if the strength of molecular force, of the particle of fluid and using vase. If the function is compared with, fx and Fx are in ratio determination independent of the distance x, putting n and moreover N, we can clearly stated that $\alpha^2 : \beta^2 = cn : CN$, i.e. the constants α^2 and β^2 are propotionals of attraction, where each distance between two molecules of equal volume, one is fluid and the other is vase. In respect to the cases of A, we denote it is acute, rectangular, obtuse and both are rectangular, as following : [36]

$$\begin{cases} \beta^2 < \frac{1}{2}\alpha^2, & \text{A is acute,} \\ \beta^2 = \frac{1}{2}\alpha^2, & \text{A is rectangular,} \\ \beta^2 > \frac{1}{2}\alpha^2 \text{ or } \beta^2 < \alpha^2 & \text{A is obtuse,} \\ \beta^2 = \alpha^2 & \text{both } \alpha \text{ and } \beta \text{ are rectangular} \end{cases}$$

: in a sense of such supposition (although there were no sufficient reasons, it looks like true, it does not contradict) it must be as follows :

- in the first case, the double quantities of particulate attractions of fluid have mutually larger than the double attractions of particle of vase of fluid ;
- in the secondary case, the quantities of first attraction were equal to the double of another ;
- in the third case, the first quantities is minor than double attractions of the other, or the first quantities are larger than another ;
- finally, in the fourth case, the quantities of both attraction are equal.

The first example explains the case of mercury in glass vase.

7.32. In the case of $\beta^2 > \alpha^2$.

- How much the value of angle A in this case, where the attraction of vase become the largest than the attraction of partial fluid mutually ?
- The imaginary value, which for $\beta^2 > \alpha^2$ the formula $\sin \frac{1}{2}A = \frac{\beta}{\alpha}$ the angle A assign, at the moment prove that the supposition in such case, non admissible.
- In fact the quality $\beta^2 > \alpha^2$, we can not consist the supposition of *limit* on the surface T with the minimal condition with respect to the function W.[37]

[36] cf. (32), (33).

[37] By (32) and (33), we get

$$\int zds - \frac{1}{2g}cs\psi_0 + \frac{1}{2g}\pi ct\theta_0 - \frac{1}{g}\pi CT\Theta_0 \quad \Rightarrow \quad \int zds - \frac{1}{2g}cs\psi_0 + \alpha^2(T + U) - 2\beta^2 T$$

- In everywhere, namely, it seems to be, if we considerinfinitesimal expansion as the ultra limit of the fluid layer, as well as T, we take the argument T', and as well as U, to which this argument approximately equals, the value of function W assume the sensible variation equals to negative quality $-(2\beta^2 - 2\alpha^2)T'$; this value W continues decreasing infinitesimally for a long time, would occupy total surface of vase up to T'.
- Variation $-(2\beta^2 - 2\alpha^2)T'$ the more it becomes exact, the more the thickness takes minor, and as long as we discuss the value of expression of W, nothing disturb, these thickness takes continuing to disappearance.
- However, this disappearing thickness (exactly distinguish with insensible) is exists except for the mathematical fiction, so that the minimum value for W is got in the case of $\beta^2 = \alpha^2$.
- However, we change the view the in problem of our phisics, when the following accesary naturally this thickness must have pleasure, even if insensible, that it can consist equilibrium.
- Whenever this part approaches, the expression of W, such as we have mentioned in the art. 18, it is imcomplete, and we denote it the part of vase, which the layer covers by T', whose thickness in the point is indefine by ρ, the expression Ω[38] extends moreover the boundary

$$\pi c^2 \int \theta'\rho.dT' - \pi cC \int \Theta'\rho.dT'$$

Until this time, the value of this W,

$$\frac{\pi C}{g} \int \Theta'\rho.dT' - \frac{\pi c}{g} \int \theta'\rho.dT' = \int dT' \Big(\frac{2\beta^2}{\Theta_0}.\Theta'\rho - \frac{2\alpha^2}{\theta_0}.\theta'\rho \Big) \tag{60}$$

where, we substitute (60) by the terms as we had denoted in (32) and (33) as follows :

$$\alpha^2 \equiv \frac{\pi c\theta_0}{2g}, \quad \beta^2 \equiv \frac{\pi CT\Theta_0}{2g}, \quad t \equiv T + U,$$

- Therefore, the value of this W, by entension of such a layer, then accept the variation $2(\beta^2 - \alpha^2)T'$, the total variation, its value of W, which we have the situation of the layer omitted, then we have

$$-2 \int dT' \Big[\beta^2 (1 - \frac{\Theta'\rho}{\Theta_0}) - \alpha^2 (1 - \frac{\theta'\rho}{\theta_0}) \Big]$$

This variation, for $\theta'_0 = \theta_0$ and $\Theta'_0 = \Theta_0$, become zero for disappearance of thickness : $\theta'\rho$ and $\Theta'\rho$ reduce the density of ρ, the thickness decrease, and then for insensible value of this ρ, evaluated as insensible, the variation of thickness inverse the value $-2(\beta^2 - \alpha^2)T'$ converges, moreover for the equilibrium state of fluid, the expression W becomes never suitable correctly if ultra sensibly decrease, it turns equivalently into sensible.

$$\int zds - 2(\beta^2 - \alpha^2)(T + T') - \alpha^2 T + \alpha^2 U$$

If $\beta^2 - \alpha^2 = 0$ then

$$\int zds - \alpha^2 T + \alpha^2 U$$

i.e. which expression, in the minimum, become for the case $\beta^2 = \alpha^2$.

- Hence, we get the figure of equilibrium fluid in vase, as $\beta^2 > \alpha^2$, for brevity, as the figure of equilibrium fluid in vase $\beta^2 = \alpha^2$, here the difference is strict equilibrium results in the layer of the insensible thickness.
- Besides, Mr. Laplace then stated that, for this case of vase of fluid insensible thickness are covered equivalent to be strictly with such vase, whose particles, the attractive force of fluid particles exist mutually and uniformly.

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then we get (33) :

$$W \equiv \int zds + (\alpha^2 - 2\beta^2)T + \alpha^2 U$$

[38] c.f. (25):

$$\Omega = -gc \int zds + \frac{1}{2}c^2 \iint ds.ds'.\varphi(ds, ds') + cC \iint ds.dS.\Phi(ds, dS)$$

- By itself, hence, the arrangement obeys the descriptions in the art. 18 read as the vertival capillarity ascending fluid in tube : quantity clearly $\beta^2 > \alpha^2$, in which we proposed the formulae that can substitute β with α in this point.

7.33. In the case of $\beta^2 < \alpha^2$.

- In this case, where $\beta^2 < \alpha^2$, the wet vase with the insensible fluid layer can not have the point, even if law of function θ' and Θ' are, when for the value of the function

$$\alpha^2(1 - \frac{\theta'\rho}{\theta'_0}) - \beta^2(1 - \frac{\Theta'\rho}{\Theta'_0})$$

for brevity, we describe as $Q\rho$, this value continues increasing, if ρ increases from the sensible value at the zero value : because, clearly from the characteristic of this function $Q\rho$ would contradict with minimal condition.
- By itself, this characteristic occures the phypothesis, by that in the article 31, where we had stated about fx and Fx are determinated independently in propotion of x, from this fact, we deduce that $\frac{\theta'\rho}{\theta'_0} = \frac{\Theta'\rho}{\Theta'_0}$, and namely, $Q\rho = (\alpha^2 - \beta^2)(1 - \frac{\theta'\rho}{\theta'_0})$.
- However, if the functions f and F will occur simultaneously as inverse, it is not at all impossible, that this value $\frac{\theta'\rho}{\theta'_0}$ rapidly decrease, as well as $\frac{\Theta'\rho}{\Theta'_0}$, the function $Q\rho$, in both insensible value of this ρ, at first negative, and after, their values reachs to minimum (i.e. at last, negative), while $\alpha^2 - \beta^2$ ascends by the value 0 of the inverse their positive limit.
- In this case equilibrium at least postulate with insensiblity, this thickness in general, showing is stated such that $Q\rho$ contradict not at all sensiblly with the least value.
- Although if we denote by $-(\beta')^2$, it turns to $(\beta')^2 < \beta^2$; the figure of other part of the substantial fluid without determinated, moreover, if in vase, with respect to the situation, β^2 must substitute the quantity $(\beta')^2$, i.e. the angle between plane of the free surface of fluid in contacting substantial part tangent with the wall of vase turns into 2 $\arcsin\frac{\beta'}{\alpha'}$. (cf. (59).)
- Moreover, doubts in such case existing in natural phenomena, seem to be filled with the more complicated phenomena.

7.34. Summary.

Another with our proposition we presented, the general principle of this sort of stability descending as a result of special phenomena, especially, essential principles fit the theory in this case, by Mr. Laplace and the contemporary with him rushed and succeded, so many phenomina in fluid equilibrium were solved, the new and so many results were produced : however, even so, the reserved were remained. Inversely, from this, it is possible to indulge in giving out the new light of this argument, or to fall into incorrect interpretation.

¶ I.

- Our theory does not only arrogate by ourself to determinate the figure of fluid equilibrium in mathematical exactitude, but also we recognize that, of the determination of figure, as follows : equilibrium figure varies different only in sensible quantity.
- If we recognize that there are errors in theory something imperfect, then they were
  - to prove in total, or,
  - to prove how much it is possible, or,
  - to prove how long we ignore the molecular attraction.
- In state of equilibrium, the function Ω [39] becomes exactly maximun, so that, the function

$$\frac{2\pi cs\psi 0}{g} - \frac{\Omega}{gc}$$

becomes minimum, this, moreover, for the indole of the molecular attraction, not only the function W is the exact equation, nevertheless, but also insensible in this place different.
- Figure for this W fit minimum, not exact equilibrium figure, if differential become insensible, as long as everywhere move sensible, the function W becomes lowest in the value of figure.
- Clearly, sensible differential in surface curvature is not excluded, as long as it were limited by partially insensible surface :

[39] c.f. (25).

  - because in equlibrium figure, exact constant-angle over A denotes impossible by considering it sufficient, that if there were immensurable distance between the vase, as Mr. Laplace then had thought correctly that, as if the inclination in limit of sphere of sensible attraction between vase is coincident with sensible value of A.

¶ II.

- We should clearly distinguish the equilibrium figure with quiet figure. Fluid equation in the state of equilibrium, it keeps. In the quiet figure of fluid have a little different equilibrium figure, nevertheless, can occure, and fluid in quiet permanent or if moving, accept the momentum in this moment, before reaching to the equilibrium of fluid, samely, for example, cubic horizontal plane not only in equilibrium but also super plane.
- Clearly, the first fundamental equation (§28) independently of perturvative limit P, i.e. in addition to, not only minimum condition but also necessary condition, here, we suppose this invariable limit : why, how long this perfect fluid delights in flow, on the other hand, at the same time, another fluid is able to increase freedom, while we postulate the minimum force of motion, the fluid will accomodate inevitably by itself its condition.
- The second principlal reason (§29) essentially depend on perfect limit of P on the surface of vase.
- Minimum condition in value W in itself we postulate the equation $i = A$: in fact, since surface fluid will accomodate this first principle in itself, the angle i does not yet reach the normal value, the value is not only W absolute minimum, but also in the equilibrium state, it can not become perfect without translocation of limit P if without fluid motion in contact with vase, what sort of motion can inevitably obstacle friction.
- From here, it is clear that, in an experiment, why each corps institutes this great differential would meet with the angular value i .
- Samely, in the case, where, $\beta^2 > \alpha^2$, the fluid in vase, whose wall get wet at this time, above all, which is consisted of the law of equilibrium, next, in part, which is substantial fluid, become $i = 2\pi$:
  if this wall in vase were dry until now except for fluid, which is in the state of non equilibrium base of dry vase rised to be possible to become quiet, after that, the value of angle i reach to 2π.
- From here, on the other hand, the theory tells us that the capillar phenomena of fluid, such that including the wet wall,
  - in the dry tube, this shows many irregularities, ascending very frequently, small by far,
  - in the wet tube at this time, where the most beautiful harmony with theory is always seen.

¶ III.

The constant inequality made by α and β, from the phenomena it is deduced,

- when the inequality becomes $\beta > \alpha$: where, the figure whose fluid in vase forms equilibrium of various material by its case not defferent with respect to immensurable vase got wet.
- Another inequality $\beta < \alpha$: where, it determinates the ratio inter the constant which is the aide of the angle i, therefore, when the mode of ratio that the force is estimate as scarecely.
- On mercury in the glass vase, Mr. Laplace studied the angle to be $i = 43°12'$.
- Wide of large precision by far, is determinated the constant α, especially if the wet vase can admit so.
- For water, at 8.5° C in temperature, we should determinate according to the experience cited by Mr. Laplace. [40]
- These sorts of things were already studied by phisicians Segner and Gay-Lussac :

8. Effects of Gauss' work

The famous expert of calculas of variations, Bolza [2] stated the situation after Gauss' [10] as follows :

> Bald nach der Veröffentlichung der Gaußschen Arbeit (1830) setzen eine Reihe von umfangreichen und wichtigen Arbeiten über die allgemeine Theorie der Extrema von Doppel- und mehrfachen Integralen ein, die wir zum größten Teil bereits im Vorhergehenden kurz erwähnt haben : Bordoni (1831), Poisson(1833 ; datiert 10. November 1831), Pagni (1835 ; datiert 15. Dezember 1834), Ostrogradsky (1836 ; datiert 24. Januar

[40] Followings are the footnote by Gauss : H denoted by Mr. Laplace corresponds to our $\pi c\theta 0$, since we denotes α in the author's expressin (32), then the expression $\frac{g}{\pi c\theta 0}$ equals to $\frac{1}{2\alpha^2}$.

1834), Delauny (1843 ; eingereicht vor 1. April 1842), Cauchy (1844), Sarrus (1848 ; eingereicht vor 1. April 1842), Lindelöf (1856). [2, p.42]

(Trans.) Soon after publishing of Gauss' paper (1830), a field of widen and important researches of the general extremal theory [41] with triple and multiple integral was started, in which we get many papers in above-mentioned shortly such as : Bordoni (1831), Poisson (1833 ; dated 10/November 1831), $\cdots$, Dalauny (1843 ; proposed 1/April 1842), $\cdots$.

9. Poisson's paper of capillarity [31]

9.1. Poisson's comments on Gauss [10].

Poisson [31] commented in the preface about Gauss [10]:

- Gauss' success is due to the merit of his ≺ *characteristic* ≻
- even Gauss uses the same method as the given physics by Laplace.
- Gauss calculates by the condition only the same density and incompressibility

After all, Poisson insists that

- We can take even any method to solve the problem, and carefully check our own equations and conditions from every points.

The following is a paragraph of the preface by Poisson[31] :

> Par les règles connues du calcul des variations, on détermine la surface inconne du liquide qui rend cette somme un *minimum*, et, comme on sait, on trove à la fois l'équation gènèrale de cette surface et l'équation particulière de son contour, ce qui est l'avantage ≺ caractèristique ≻ de la méthode que M.Gauss a suive. Mais cet illustre géomètre étant parti des mêmes donnèes physiques que Laplace, et n'ayant pas non plus considéré la variation de densité aux extrèmités du liquide, qu'il a regardé, au contraire, comme incompressible dans tous ses parties, les objections qui s'élèvent contre la théorie de l'autre que par la manière de former les équations d'équilibre. On peut, á cet égard, employer différens moyens ; mais, sans craindre de compliquer le calcul et d'en augmenter les difficultés, il importe de ne négliger aucune des circonstances essentielles de la question, parmi lesquelles il faut compter surtout la dilation du liquide près de sa surface libre et la condensation qui peut être produite par l'attraction du tube. [31, 8]

(Trans.) By the method known as calculas of variations, we determine the unknown surface of fluid which this sum show munimum, and as we know, we get at once the genaral equation of the surface and the paticular equation of the arbitrary height, these are due to the characteristic advantages of the method Mr. Gauss had approached. But even this great prodigious mathematecian had based the samely given phisycs with Laplace, and not considering the variation of density at the extremity of liquid, where there is regard contraly, as the incompressibe in all the particle, the objection which evolutes to another theory than by the manner of formulation of the equilibrium equations. We can, in this point, use the different methods; but without being afraid to the calculation and the difficulties extended by it, it is important not to neglect any essential circumstances of the problems, among which, to challenge especially the dilation of liquid in neighbourhood of free surface and condensation producing by the attraction of tube.

9.2. Poisson's two constants : K and H.

We cite Poisson's K and H from [31, 12-14].

$$K = 2\pi\rho^2 q \int_0^\infty r^3 \varphi r dr$$

where,

$$q \equiv \int_0^\infty \int_0^\infty \frac{(y+z)dydz}{[1+(y+z)^2]^{\frac{3}{2}}} = \frac{1}{3}\int_0^\infty \frac{dy}{(1+y^2)^{\frac{3}{2}}} = \frac{1}{3}$$

$$(1)_P \quad K = \frac{2}{3}\pi\rho^2 \int_0^\infty r^3 \varphi r \, dr \tag{61}$$

$$\eta = u \sin v, \quad \eta' = \cos v$$

[41] This term is called extremum (*pl.* extrema) which includes maximum and minimum.

$$\zeta = Q\eta^2 + Q'(\eta')^2 + Q''\eta\eta'$$

We denote λ and λ' radii of two principle curvatures.

$$\frac{1}{\lambda} = \frac{d\zeta}{d\eta^2} = 2Q, \quad \frac{1}{\lambda'} = \frac{d\zeta}{d(\eta')^2} = 2Q',$$

The average value

$$\mu = -H(Q+Q') = -\frac{1}{2}H\Big(\frac{1}{\lambda} + \frac{1}{\lambda'}\Big),$$

where, we denote H for convienience sake

$$H \equiv \pi\rho^2 \int_0^\infty \int_0^\infty \varphi r \frac{su^3}{r} duds$$

where,

$$s = ux, \quad ds = udx, \quad u = \frac{r}{\sqrt{1+x^2}}, \quad du = \frac{dr}{\sqrt{1+x^2}}$$

$$(2)_P \quad H = \pi\rho^2 \int_0^\infty r^4 \varphi r dr \int_0^\infty \frac{xdx}{\sqrt{1+x^2}} = \frac{1}{4}\pi\rho^2 \int_0^\infty r^4\varphi r\ dr \tag{62}$$

The normal action on this point :

$$(3)_P \quad N = K - \frac{1}{2}H\Big(\frac{1}{\lambda} + \frac{1}{\lambda'}\Big) \tag{63}$$

9.3. Coincidence of Poisson's K and H with Laplace's K and H.

Poisson proved Laplace's formulae as follows :

Les expressions des coefficiens K et H que cette formule renferme s'accordent avec celles que Laplace a trouvées, sous une autre forme, pour les mêmes quantités. En effet, on suppose, dans la *Mécanique céleste*,[42]

$$\int \varphi r\ dr = c - \Pi r, \quad \int r\Pi r\ dr = c' - \Psi r \tag{64}$$

les intégrales commençant avec r, c et c' étant leurs values quand r a une grandeur sensible, Πr et Ψr désignant des founctions qui s'évanouissent pour tout valeur sensible de r. D'apprés cela, on a

$$K = 2\pi\rho^2 \int_0^h \Psi r\ dr, \quad H = 2\pi\rho^2 \int_0^h r\Psi r\ dr \tag{65}$$

en rétablissant la densité ρ que Laplace a prise pour unité, et la limite h étant une quantité de grandeur sensible, qu'on pourra, si l'on vent, remplacer par l'infini. Or, si l'on intègre par partie, il vient

$$K = 2\pi\rho^2 h\Psi h - 2\pi\rho^2 \int_0^h r\frac{d\Psi r}{dr} dr = 2\pi\rho^2 \int_0^h r^2 \Pi r\ dr,$$

$$H = \pi\rho^2 h^2 \Psi h - \pi\rho^2 \int_0^h r^2 \frac{d\Psi r}{dr} dr = \pi\rho^2 \int_0^h r^3 \Pi r\ dr$$

intégrant de nouveau, on a

$$K = \frac{2\pi\rho^2}{3} h^3 \Pi h - \frac{2\pi\rho^2}{3} \int_0^h r^3 \frac{d\Pi r}{dr} dr = \frac{2\pi\rho^2}{3} \int_0^h r^3 \varphi r\ dr \tag{66}$$

$$H = \frac{\pi\rho^2}{4} h^4 \Pi h - \frac{\pi\rho^2}{4} \int_0^h r^4 \frac{d\Pi r}{dr} dr = \frac{\pi\rho^2}{4} \int_0^h r^4 \varphi r\ dr \tag{67}$$

ce qui coïncide avec les formes (61) et (62), en prenant $h = \infty$. [31, pp.14-15] .

[42] cf. (6).

TABLE 4. C_1, C_2 in capillarity

no	paper	C_1, C_2 using Ψ	C_1, C_2 using φ	N : normal action
1	Laplace 1805 [17]	$K \equiv 2\pi \int \Psi f.df,$ $H \equiv 2\pi \int \Psi f.f.df,$ (11)	$\int \varphi r\ dr = c - \Pi r$ $\Rightarrow \quad \varphi r = -\frac{d\Pi r}{dr},$ $\int r\Pi r\ dr = c' - \Psi r$ $\Rightarrow \quad r\Pi r = -r\frac{d\Psi r}{dr}$ (64) $H \equiv \frac{\pi}{4}\int_0^\infty f^4 df \varphi(f)$ $= 2\pi \int_0^\infty f df.\Psi(f),$ (9)	the action of corps on the canal beco $N = K - \frac{H}{2}.\left(\frac{1}{R} + \frac{1}{R'}\right),$ (4)
2	Gauss 1830 [10]	In citing of Laplace (11) : $\int_x^\infty \varphi f.df = \Pi x,$ $\int_x^\infty \Pi f.f df = \Psi x,$ $K \equiv 2\pi \int_0^\infty \Psi f.df,$ $H \equiv 2\pi \int_0^\infty \Psi f.f.df,$ (21)	$-fx.dx = d\varphi x,$ $\Rightarrow \int fx.dx = -\varphi x,$ (23) $-Fx.dx = d\Phi x,$ $\Rightarrow \int Fx.dx = -\Phi x,$ (24)	*Remark* : Gauss says : "In the calculas (11) by Mr. Laplace, we have at least a thing, which we can give evidence about it, and for which we would not absolutely consent with him." (cf. § 7.0 Preface by Gauss [10].)
3	Poisson 1830 [31]	$K \equiv 2\pi\rho^2 \int_0^h \Psi r\ dr,$ $H \equiv 2\pi\rho^2 \int_0^h r\Psi r\ dr,$ (65)	$(1)_P \quad K \equiv \frac{2}{3}\pi\rho^2 \int_0^\infty r^3 \varphi r\ dr,$ (61) $(2)_P \quad H \equiv \frac{1}{4}\pi\rho^2 \int_0^\infty r^4 \varphi r\ dr,$ (62) $K \equiv \frac{2\pi\rho^2}{3} h^3 \Pi h - \frac{2\pi\rho^2}{3} \int_0^h r^3 \frac{d\Pi r}{dr} dr$ $= \frac{2\pi\rho^2}{3} \int_0^h r^3 \varphi r\ dr,$ (66) $H \equiv \frac{\pi\rho^2}{4} h^4 \Pi h - \frac{\pi\rho^2}{4} \int_0^h r^4 \frac{d\Pi r}{dr} dr$ $= \frac{\pi\rho^2}{4} \int_0^h r^4 \varphi r\ dr,$ (67)	$(3)_P \quad N = K - \frac{1}{2} H\left(\frac{1}{\lambda} + \frac{1}{\lambda'}\right),$ (63) *Remark* : If $h = \infty$ and density $\rho = 1$, then (65) equals to Laplace (11), and (67) with (9) respectively.

(Trans.) The expressions with coefficients K and H which these formulae included are coincident with that which Laplace had found under another form of (65), for the same quantities. In fact, we see that in *Mécanique céleste,* as follows :

$\cdots$ (65)

the integrals of the right hand-side of (64) beginning with r, c, and c' in which values r were sensiblly large, Πr and Ψr are designated as *the evaporating functions*, even if r were sensiblly large value. For this reason, it turns : [43]

$\cdots$ (expressions)

Laplace set density by $\rho = 1$, and h the big value, then we substitute h with ∞. Or if integrate by partial, it turns

$\cdots$ (66), (67)

where if we replace $h = \infty$, then we get a coincidence with our formulae (61) and (62). [44]

10. The characteristic δ by Green

Green[12] says in 1828 : Hence, the integral

$$\iiint dxdydz \left\{ \frac{dV}{dx}\frac{dU}{dx} + \frac{dV}{dy}\frac{dU}{dy} + \frac{dV}{dz}\frac{dU}{dz} \right\},$$

by using the $\prec$ *characteristic* δ $\succ$ in order to abridge the expression, becomes

$$-\int d\sigma V \frac{dU}{dw} - \iiint dxdydz V \delta U.$$

where, dw being an infinitely small line perpendicular to the surface, and measured from this surface towards the interior of the body. [12, 23-25]

11. Conclusions

- We can get the common thinking-methods for handling the fluid and formulating the microscopically-descriptive equations in each type of equation such as
  (1) hydrostatics started by Clairaut[6], and evolved by Laplace, Gauss to Poisson who formulated capillarity equations,
  (2) Navier and Poisson who formulated the equilibrium equations,
  (3) kinetic hydrodynamics by Navier, Cauchy, Poisson, Saint-Venant and Stokes who formulated the Navier-Stokes equations.
- Gauss criticised Laplace's two-constants K and H, however, they are coincident with Poisson's one. (cf. Table 4.)
- After Gauss, the methods of integration and calculas of variations were improved rapidly, since he had solved the bottleneck due to Gauss' formula of integral as O.Bolza [2] stated.

12. Acknowledgements

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13. Errata of last I.M.C.T.C Report

We would like to correct the description of equations by Navier (9), (10), (11), (12) and (13) in "*Saint-Venant and Navier-Stokes equations - Report of 19th Symposium on the History of Mathematics,*" published by The Institute for Mathematics and Computer science of Tsuda College **30**, [23, pp.9-66], in which we should delete plus signs in pp.17-18. We are sincerely sorry that we had not correctly reflected the equations by Navier [26]. We show its equations corrected as follows :

• ¶ 5.

Errors (plus signs existing in the 2nd and 3rd lows of the tensor part):

$$(5\text{-}1)_{N^e} \quad 0 = \varepsilon \iiint da\, db\, dc \left\{ \begin{array}{l} 3\frac{dx}{da}\frac{\delta dx}{da} + \frac{dx}{db}\frac{\delta dx}{db} + \frac{dx}{db}\frac{\delta dy}{da} + \frac{dy}{da}\frac{\delta dx}{db} + \frac{dy}{da}\frac{\delta dy}{da} + \frac{dx}{da}\frac{\delta dy}{db} + \frac{dy}{db}\frac{\delta dx}{da} \\ +\frac{dx}{dc}\frac{\delta dx}{dc} + \frac{dx}{dc}\frac{\delta dz}{da} + \frac{dz}{da}\frac{\delta dx}{dc} + \frac{dz}{da}\frac{\delta dz}{da} + \frac{dx}{da}\frac{\delta dz}{dc} + \frac{dz}{dc}\frac{\delta dx}{da} + 3\frac{dy}{db}\frac{\delta dy}{db} \\ +\frac{dy}{dc}\frac{\delta dy}{dc} + \frac{dy}{dc}\frac{\delta dz}{db} + \frac{dz}{db}\frac{\delta dy}{dc} + \frac{dz}{db}\frac{\delta dz}{db} + \frac{dz}{dc}\frac{\delta dy}{db} + \frac{dy}{db}\frac{\delta dz}{dc} + 3\frac{dz}{dc}\frac{\delta dz}{dc} \end{array} \right.$$

$$- \iiint da\, db\, dc\, (X\delta x + Y\delta y + Z\delta z) - \int ds\, (X'\delta x' + Y'\delta y' + Z'\delta z'). \qquad (9)$$

Corrected (only deleting plus signs):

$$(5\text{-}1)_{N^e} \quad 0 = \varepsilon \iiint da\, db\, dc \begin{cases} 3\frac{dx}{da}\frac{\delta dx}{da} + \frac{dx}{db}\frac{\delta dx}{db} + \frac{dx}{db}\frac{\delta dy}{da} + \frac{dy}{da}\frac{\delta dx}{db} + \frac{dy}{da}\frac{\delta dy}{da} + \frac{dx}{da}\frac{\delta dy}{db} + \frac{dy}{db}\frac{\delta dx}{da} \\ \frac{dx}{dc}\frac{\delta dx}{dc} + \frac{dx}{dc}\frac{\delta dz}{da} + \frac{dz}{da}\frac{\delta dx}{dc} + \frac{dz}{da}\frac{\delta dz}{da} + \frac{dx}{da}\frac{\delta dz}{dc} + \frac{dz}{dc}\frac{\delta dx}{da} + 3\frac{dy}{db}\frac{\delta dy}{db} \\ \frac{dy}{dc}\frac{\delta dy}{dc} + \frac{dy}{dc}\frac{\delta dz}{db} + \frac{dz}{db}\frac{\delta dy}{dc} + \frac{dz}{db}\frac{\delta dz}{db} + \frac{dz}{dc}\frac{\delta dy}{db} + \frac{dy}{db}\frac{\delta dz}{dc} + 3\frac{dz}{dc}\frac{\delta dz}{dc} \end{cases}$$

$$- \iiint da\, db\, dc\, (X\delta x + Y\delta y + Z\delta z) - \int ds\, (X'\delta x' + Y'\delta y' + Z'\delta z'). \qquad (9)$$

We show the corrected (10), (11),(12) and (13) as well as (9) as follows :

When the first term of ε in the right-hand side of (9) is arranged in respect to $\delta x, \delta y$ and δz then :

$$\varepsilon \iiint dadbdc \begin{cases} 3\frac{dx}{da}\frac{\delta dx}{da} + \frac{dx}{db}\frac{\delta dx}{db} + \frac{dx}{dc}\frac{\delta dx}{dc} + \frac{dy}{da}\frac{\delta dx}{db} + \frac{dy}{db}\frac{\delta dx}{da} + \frac{dz}{da}\frac{\delta dx}{dc} + \frac{dz}{dc}\frac{\delta dx}{da} \\ \frac{dy}{da}\frac{\delta dy}{da} + 3\frac{dy}{db}\frac{\delta dy}{db} + \frac{dy}{dc}\frac{\delta dy}{dc} + \frac{dx}{da}\frac{\delta dy}{db} + \frac{dx}{db}\frac{\delta dy}{da} + \frac{dz}{db}\frac{\delta dy}{dc} + \frac{dz}{dc}\frac{\delta dy}{db} \\ \frac{dz}{da}\frac{\delta dz}{da} + \frac{dz}{db}\frac{\delta dz}{db} + 3\frac{dz}{dc}\frac{\delta dz}{dc} + \frac{dx}{da}\frac{\delta dz}{dc} + \frac{dx}{dc}\frac{\delta dz}{da} + \frac{dy}{db}\frac{\delta dz}{dc} + \frac{dy}{dc}\frac{\delta dz}{db} \end{cases} \qquad (10)$$

Moreover, we rearrange (10) for differential : $\frac{\delta x'}{da'}, \frac{\delta x'}{db'}, \frac{\delta x'}{dc'}, \frac{\delta y'}{da'}, \frac{\delta y'}{db'}, \frac{\delta y'}{dc'}, \frac{\delta z'}{da'}, \frac{\delta z'}{db'}, \frac{\delta z'}{dc'}$ as follows :

$$\varepsilon \iiint dadbdc \begin{cases} 3\frac{dx}{da}\frac{\delta dx}{da} + \frac{dy}{db}\frac{\delta dx}{da} + \frac{dz}{dc}\frac{\delta dx}{da} + \frac{dx}{db}\frac{\delta dx}{db} + \frac{dy}{da}\frac{\delta dx}{db} + \frac{dx}{dc}\frac{\delta dx}{dc} + \frac{dz}{da}\frac{\delta dx}{dc} \\ \frac{dx}{db}\frac{\delta dy}{da} + \frac{dy}{da}\frac{\delta dy}{da} + \frac{dx}{da}\frac{\delta dy}{db} + 3\frac{dy}{db}\frac{\delta dy}{db} + \frac{dz}{dc}\frac{\delta dy}{db} + \frac{dy}{dc}\frac{\delta dy}{dc} + \frac{dz}{db}\frac{\delta dy}{dc} \\ \frac{dx}{dc}\frac{\delta dz}{da} + \frac{dz}{da}\frac{\delta dz}{da} + \frac{dy}{dc}\frac{\delta dz}{db} + \frac{dz}{db}\frac{\delta dz}{db} + \frac{dx}{da}\frac{\delta dz}{dc} + 3\frac{dz}{dc}\frac{\delta dz}{dc} + \frac{dy}{db}\frac{\delta dz}{dc} \end{cases} \qquad (11)$$

Using (10) and partial integration of $\delta x, \delta y$ and δz, we make the top term of (12), in which a minus sign is leading. And using (11), we show only the first differential order : $\delta x', \delta y', \delta z'$ in the middle term of (12) as follows :

$$(5\text{-}2)_{N^e} \quad 0$$

$$= -\varepsilon \iiint da\, db\, dc \begin{cases} \left(3\frac{d^2x}{da^2} + \frac{d^2x}{db^2} + \frac{d^2x}{dc^2} + 2\frac{d^2y}{dadb} + 2\frac{d^2z}{dadc}\right)\delta x \\ \left(\frac{d^2y}{da^2} + 3\frac{d^2y}{db^2} + \frac{d^2y}{dc^2} + 2\frac{d^2x}{dadb} + 2\frac{d^2z}{dbdc}\right)\delta y \\ \left(\frac{d^2z}{da^2} + \frac{d^2z}{db^2} + 3\frac{d^2z}{dc^2} + 2\frac{d^2x}{dadc} + 2\frac{d^2y}{dbdc}\right)\delta z \end{cases}$$

$$+ \quad \varepsilon\Big[\iint db'\, dc'\Big(3\frac{dx'}{da'} + \frac{dy'}{db'} + \frac{dz'}{dc'}\Big) + \iint da'dc'\Big(\frac{dx'}{db'} + \frac{dy'}{da'}\Big) + \iint da'db'\Big(\frac{dx'}{dc} + \frac{dz'}{da'}\Big)\Big]\delta x'$$

$$+ \quad \varepsilon\Big[\iint db'dc'\Big(\frac{dx'}{db'} + \frac{dy'}{da'}\Big) + \iint da'dc'\Big(\frac{dx'}{da'} + 3\frac{dy'}{db'} + \frac{dz'}{dc'}\Big) + \iint da'db'\Big(\frac{dy'}{dc'} + \frac{dz'}{db'}\Big)\Big]\delta y'$$

$$+ \quad \varepsilon\Big[\iint db'dc'\Big(\frac{dx'}{dc'} + \frac{dz'}{da'}\Big) + \iint da'dc'\Big(\frac{dy'}{dc'} + \frac{dz'}{db'}\Big) + \iint da'db'\Big(\frac{dx'}{da'} + \frac{dy'}{db'} + 3\frac{dz'}{dc'}\Big)\Big]\delta z'$$

$$- \quad \iiint da\, db\, dc(X\delta x + Y\delta y + Z\delta z) - \int ds(X'\delta x' + Y'\delta y' + Z'\delta z'). \qquad (12)$$

We solve the indeterminate equations (12) [45] of equilibrium in an elastic solid as follows. At first, we get the following two equations from (12) :

- The force inside the solid corps :

$$-\iiint da\, db\, dc(X\delta x + Y\delta y + Z\delta z) = \varepsilon \iiint da\, db\, dc \begin{cases} \left(3\frac{d^2x}{da^2} + \frac{d^2x}{db^2} + \frac{d^2x}{dc^2} + 2\frac{d^2y}{dadb} + 2\frac{d^2z}{dadc}\right)\delta x \\ \left(\frac{d^2y}{da^2} + 3\frac{d^2y}{db^2} + \frac{d^2y}{dc^2} + 2\frac{d^2x}{dadb} + 2\frac{d^2z}{dbdc}\right)\delta y \\ \left(\frac{d^2z}{da^2} + \frac{d^2z}{db^2} + 3\frac{d^2z}{dc^2} + 2\frac{d^2x}{dadc} + 2\frac{d^2y}{dbdc}\right)\delta z. \end{cases} \qquad (13)$$

[45] Navier says that (12) is usually called "equations indéfinies". [25, p.384,389]