THE RECEPTION OF FREDHOLM'S RESULTS ON INTEGRAL EQUATIONS: PRELIMINARY REPORT FROM TSUDA COLLEGE SYMPOSIUM, TOKYO, 2009

R. TAZZIOLI AND T. ARCHIBALD

La théorie des équations intégrales, née d'hier, est d'ores et déjà classique. Elle a fait son entrée dans plusieurs de nos enseignements. Nul doute que – peut-être à la faveur de nouveaux perfectionnements – elle ne s'impose bientôt à la pratique courante de calcul. C'est une fortune rare parmi les doctrines mathématiques, si souvent destinées à rester des objets de musée.

J. Hadamard, Préface to [Heywood and Fréchet 1912], v.

1. Introduction and Background

1.1. Integral Equations, Functional Analysis, and Boundary Value Problems. For the mathematics student of today, the 1900 result of Ivar Fredholm (1866-1927) will typically appear in a fairly advanced introduction to analysis, in a chapter on compact operators, for example as a corollary to Atkinson's theorem. Fredholm's result in modern language states roughly that the spectrum of a compact operator T on a Hilbert space consists of $\{0\}$ and the eigenvalues for T, and is a countable subset of the complex plane with 0 as the only possible accumulation point ([Pedersen 1988], 111). This tidy description obscures the pivotal position of this theorem in the origins of functional analysis. The theorem was conceived as a contribution to the theory of functional equations involving integrals, though the author, Ivar Fredholm, saw at once its usefulness in demonstrating the existence of solutions to certain boundary value problems involving partial differential equations. In this paper we briefly explore the context in which the theorem was developed, and discuss one aspect of its reception. The solution of such boundary value problems, and the question of the existence of solutions, occupied a large number of mathematicians throughout Europe at the time, and the techniques provided by Fredholm's result garnered an enthusiastic audience. Indeed, the "integral equation method" for the study of differential equations became a standard feature of the mathematical landscape by around 1915, with the appearance of several textbooks and expository accounts of the theory as well as its inclusion in lecture courses on analysis. Hilbert's response to the theorem was deeper, and had broader consequences. By 1904, Hilbert had already grasped analogies between the study of certain cases of the Fredholm result and the theory of quadratic forms. Together

Thanks are due to Kazuaki NAGAOKA for the invitation, and to Ken Saito for his kind efforts in facilitating the visit. TA's research for this paper was supported by the Social Sciences and Humanities Research Council (Canada), and by the Dibner Institute for the History of Science and Technology, MIT..

with a number of students, notably Erhard Schmidt, Hilbert formulated the basic ideas of what are now know as Hilbert spaces and linear operators on them with decisive effect for the future of mathematics. The impact of Hilbert's work took several decades to be fully felt. While other threads fed into Hilbert's understanding of this area, the insight afforded by the Fredholm result provides the reader of today with one of the clearest points of entry into the study of the origins of functional analysis. There are a number of historical treatments which investigate this and related issues in some detail, perhaps most notably [Dieudonné 1981].

In this paper, however, we concentrate on the response within the community of researchers on differential equations in France and Italy, where Fredholm's result was understood as a fruitful and powerful method for proving the existence of solutions to boundary value problems. In contrast to the work of Hilbert and his students, this work appears conservative,and lacks the abstract, generalizing stamp which was to become a hallmark of twentieth century mathematics. Nonetheless, the reception of Fredholm's work marks an important moment in the development of research in analysis in these two countries, as we shall discuss below. Furthermore, the case provides insight into the ways in which innovative work is received in different national and institutional contexts.

This work is certainly part of the background leading to the development of functional analysis as a free-standing entity and clearly identified research specialty. However, it should not be imagined that all these researchers were consciously involved in the construction of such a research specialty, nor were they intentionally carrying out specific elements of what were to be its later research programs. It is of course true that ultimately one can see specific results from this period as special cases of functional-analytic results. But there was nothing called functional analysis at this point in time (all our discussions are limited to the period before 1915).

In what follows, we begin with an account of some background developments in the theory of differential equations. We then proceed to a discussion of Fredholm's result and its reception.

1.2. **Background: Schwarz, Neumann and the Dirichlet Problem for the Laplace equation.** Integral equations are, in a sense, nearly as old as integrals. However, for our purposes they are a nineteenth-century development. Early work by Abel and Liouville has been described well in [Lützen 1990]. Some immediate background activity related to our discussion was due to Carl Neumann and Poincaré in one direction; to Picard and Le Roux in another; and to Volterra. [1]

In the 1870s, Carl Neumann (1832-1925) devised a method for solving the Dirichlet problem for the Laplace-Poisson equation which was to be of considerable importance for many writers, notably Poincaré and Fredholm.

Recall that the Laplace-Poisson equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4\pi\rho$$

is important in gravitation ($\rho = 0$), electrostatics, and steady-state heat conduction. The Dirichlet problem involves solving this equation on a bounded domain given values for f on the boundary. As mentioned above, Neumann devised his so-called

[1]Volterra later claimed, and we have no reason to doubt it, that he had Fredholm's results in 1896. He did in fact publish a verification of a result obtained using them, but did not give the proofs, which he knew to be insufficiently rigorous. See [Tricomi 1957], 5.

"method of the arithmetic mean" in 1870 to permit series computation of a solution. More importantly for our present topic, Neumann in 1877 showed that a solution can be written in integral form where the function on the boundary is viewed as a *double-layer potential* (think of little magnetic dipoles with a variable moment):

$$\int\int \rho \frac{\cos\phi}{2\pi r^2} d\sigma$$

where in this case the function you wish to find is ρ. The integrand expresses the potential of a density ρ sitting on a boundary element $d\sigma$ on a point at distance r, and ϕ is the angle between the normal to the element and the line from the element to the point. This approach, combined with method of successive approximations, gives an existence proof for one kind of equation with fairly strong hypotheses.

1.3. **Picard and the Method of Successive Approximations.** Emile Picard (1856-1937) became involved in research in differential equations early in his career, largely through his contact with his mentor Charles Hermite. Following in Hermite's footsteps, and influenced by contemporary French work by writers such as Tannery, Flocquet, and Poincaré, Picard specialized at first in questions concerning what can be said about the nature of the solutions of a differential equation based on formal characteristics of the equation (such as the periodicity of the coefficients). By the 1880s, this had evolved into a general interest in existence issues, and around 1887 or 1888 he focussed his attention on existence results of H. A. Schwarz and Carl Neumann from the 1870s. The work of Schwarz and Neumann was intended to provide solutions to the Dirichlet problem, partly to replace Riemann's use of the Dirichlet principle. Both were iterative series methods, and it may be that this inspired Picard to devise his method of successive approximations, a constructive existence method for proving existence of solutions, widely applicable to both ordinary and partial differential equations.

As a trivial example, consider:

$$\frac{dx}{dt} = g(x, t), x(0) = 0$$

Let $x_{n+1}(t) = \int_0^t g(\tau, x_n(\tau))d\tau$. If this converges, it converges to a local solution, which under certain conditions may be extended to a global solution. In this case, the series that results is easily identifiable.

Picard established conditions for local convergence in the case of the 2-dimensional Laplace-Poisson equation, and uses Schwarz's procedure to assemble local solutions, proving global existence. In so doing, he shows his mastery of the Schwarz-Weierstrass language for analysis. These results were the most powerful existence methods available at the time, and rapidly became part of the standard repertoire. As Lützen has pointed out, the basic method was already known to Liouville, but Picard's discovery appears thoroughly independent.

1.4. **Poincaré.** As in so many other areas, Henri Poincaré made fundamental contributions to the field of partial differential equations which were of immediate and long-term consequence. His interest in the field dated already to his doctoral thesis, on the functions defined by partial differential equations, a difficult work which was received without much understanding. In two astonishingly rich papers of 1890 and 1894 he created a variety of tools and approaches which have had tremendous influence [Poincaré 1890],[Poincaré 1894].

In 1890, he gave the first complete proof of the existence and uniqueness of solutions to the Laplace equation with continuous boundary conditions, for a large class of three-dimensional regions. Where Neumann had defined a sequence of functions satisfying the Laplace equation, converging to one with the correct boundary condition, Poincaré instead employed a sequence of functions which aren't harmonic, but have the right boundary values, and devised a method to make the sequence converge to a harmonic function via the method of "balayage" (sweeping). He first showed that if such a Dirichlet problem for the Laplace equation can be solved when the values on the boundary are given by a polynomial in three variables, it can be solved when they are given by any continuous function. To solve the problem when the boundary value is a polynomial p, Poincaré defined a countable covering of the interior of the region by spheres $S_1, S_2, \ldots$, and used the known solution for the Dirichlet problem on a sphere to replace p by the harmonic function f given by this solution. A new function f_1 is now defined, equal to f inside the first sphere, and equal to p elsewhere. Proceeding to the second sphere, we likewise "sweep" it by solving the Dirichlet problem, likewise getting a function f_2 which satisfies the Laplace equation inside S_2 and is equal to p elsewhere. We now need to go back to the first sphere, so we continue this process in the order $S_1, S_2, S_1, S_2, S_3, \ldots$, passing through each sphere infinitely often while retaining the boundary values for the region. Poincaré was able to show that this process leads to a function with the correct boundary values which is harmonic in the entire interior of the sphere.

In the same paper of 1890, Poincaré began to look at eigenvalues. H. A. Schwarz and Picard had found the first and second eigenvalues of the Laplace operator for Dirichlet boundary conditions in 1885 and 1893 respectively. Poincaré, in 1894, found the infinite sequence of eigenvalues and their corresponding eigenfunctions, probably soon after reading of Schwarz's work in Picard's paper. This is the beginning of spectral theory, a fundamental tool of functional analysis in the twentieth century. A fine account of this work is to be found in [Dieudonné 1981].

We mention two other innovations due to Poincaré. One of these is the so-called continuity method. In 1898, Poincaré had used Picard's successive approximation method to obtain a solution for the equation $\Delta u = e^u$. Here he had the idea of approaching the solution of other non-linear equations by starting with a problem with known solution and then continuing the existence result along a parameter to a more complicated equation. This method was to some degree foreshadowed in the 1890 paper, as was a second important tool, that of the *a priori* estimate of which good use was soon to be made by S. Bernstein (see below).

Poincaré also built on Carl Neumann's work in a way which is part of the immediate background to Fredholm, Hilbert, and Picard, in particular with two papers [Poincaré 1894], [Poincaré 1897]. In these papers, the notions of eigenvalues and eigenfunctions for a particular problem are introduced, and termed "valeurs fondamentales" and "fonctions fondamentales". Poincaré obtains improved hypotheses over Neumann, getting rid of convexity, for example. The beginning of the paper is scrupulously rigorous, but changes gears in the middle, where he states "Jusqu'ici j'ai cherché à être parfaitement rigoureuse," and then goes on to discuss in nonrigorous terms the fact that one can construct an infinite series of real eigenvalues (parameter values for which there is a solution to a DE expressed as an integral using Neumann's method). The use of parameter we will see below in Fredholm's work. It is related to analytic continuation issues, later important for Bernstein as well in his work on Hilbert's nineteenth problem. Poincaré notes the possibility of eigenfunction expansions, but can't prove it, remarking: "une fois que l'on connaîtrait les fonctions fondamentales, il serait aisé de résoudre le problème de Dirichlet".

1.5. **Volterra.**

2. FREDHOLM

This brings us to Fredholm.

Fredholm's letter if Aug 8 1899 to Mittag-Leffler announces a method for solving integral equations, seen as functional equations. Fredholm gave an immediate application to the proof of existence theorems for the solution of boundary-value problems . This was published 1900 in Swedish, but communicated to Poincaré already in Dec. 1899. A French summary was published 1902 in *Comptes Rendus*, with the full version appearing in 1903 *Acta Mathematica*, in a volume in honour of Abel.

Fredholm noted that Neumann's "double-layer" method had shown how the solution to the Dirichlet problem for the Laplace-Poisson equation in two or three dimensions could be expressed as an integral, which Neumann could then find using series. Poincaré had extended Neumann's method (1894, 1896), improving the hypotheses.

Fredholm in turn considered the functional equation

$$\phi(x) + \lambda \int_0^1 f(x, y)\phi(y)dy = \psi(x)$$

where we are solving for ϕ. The resemblance to the problems considered by Neumann and Poincaré is obvious - the λ is Poincaré's parameter.

Fredholm noted:

> Most problems of mathematical physics which lead to linear differential equations are translated into functional equations [of this form, possibly with more variables].

Fredholm's basic insight consisted of the following. In the equation

$$\phi(x) + \lambda \int_0^1 f(x, y)\phi(y)dy = \psi(x)$$

we may consider the analogy with a system of linear equations with ϕ as the variable. One can then get a kind of analogy with Cramer's rule, with "determinants" and "minors" expressed as series expansions in the parameter λ, the coefficients involving functional determinants of the kernel $f(x, y)$. The expression is easily seen to formally satisfy the integral equation; convergence of the series is guaranteed by an 1893 result of Hadamard (rediscovered independently by Fredholm). The singularities of the determinantal expression are what we would now term the eigenvalues of the corresponding boundary value problem. In this regard, Fredholm understood many properties, multiplicities, etc but lacked the general viewpoint which we associate with a linear-algebraic way of addressing the subject.

Fredholm's own account uses an unorthodox notation, and for an understanding of the reception it will be useful to look at this in some detail, following it with a discussion of its modern meaning using the language of step functions and Lebesgue integrals. The account we give of the modern work is based on that of F. G. Tricomi ([Tricomi 1957]3-5, 64-75).

In [Fredholm 1902], Fredholm began by noting that, in the two-variable case, one can write many problems of mathematical physics in the form

$$\phi(x) + \int_0^1 f(x,y)\phi(y)dy = \psi(x), \quad 0 \leq x \leq 1 \tag{2.1}$$

and that the left side may be denoted $A_f\phi(x)$ for brevity. The idea of an integral as a transformation was not new but we note its explicit use here. He then further explicitly notes that the equation 2.1 is "a limiting case of the theory of linear equations," in which we have "all the results of the theory of determinants". [2]. Both the analogy with linear equations and the concept of determinant employed in the infinitary setting are left implicit by Fredholm, and we will explain the analogy.

Fredholm then considers (without stating it clearly) a partition of the unit square given by

$$0 < x_1 < x_2 < \cdots < x_n \leq 1, \quad 0 < y_1 < y_2 < \cdots < y_n \leq 1$$

and denotes the determinant of the n^2 quantities $f(x_i, y_k)$ by

$$f\begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ y_1 & y_2 & \ldots & y_n \end{pmatrix}.$$

He defines the expression he will use as the determinant for the integral equation as

$$D_f = \sum_{n=1}^{\infty} \frac{1}{n!} \int\int\cdots\int f\begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ y_1 & y_2 & \ldots & y_n \end{pmatrix} dx_1 dx_2 \ldots dx_n,$$

which is defined to be equal to 1 for $n = 0$. He then further defines k-th order minors of this expression,

$$D_f\begin{pmatrix} \xi_1 & \xi_2 & \ldots & \xi_n \\ \eta_1 & \eta_2 & \ldots & \eta_n \end{pmatrix} = \sum_{n=1}^{\infty} \frac{1}{k!} \int\int\cdots\int f\begin{pmatrix} \xi_1 & \xi_2 & \ldots & \xi_n, & x_1 & \ldots & x_n \\ \eta_1 & \eta_2 & \ldots & \eta_n, & y_1 & \ldots & y_n \end{pmatrix} dx_1 dx_2 \ldots$$

All integrals in these expressions are taken from 0 to 1.

If we can accept for a moment that these expressions actually correspond to the determinant and minors of a linear system, and that the series converge, then the next statement by Fredholm is unsurprising. Calling on Cramer's rule, if $D_f \neq 0$, we get as unique solution

$$\phi(x) = A_g\psi(x)$$

where the kernel g is given by

$$g(x,y) = -\frac{D_f\begin{pmatrix} x \\ y \end{pmatrix}}{D_f}.$$

An account of the validity of this result is given in Appendix 1.

[2] "un cas limite de la théorie des équations linéaires...tous les résultats de la théorie des déterminants."[Fredholm 1902], 219)

3. Reception of Fredholm's Theory to 1906

3.1. Hilbert's School, Partial Differential Equations, and the Fredholm Theory. The response to Fredholm's result was electric. Most writers seized on it as a method for solving classes of boundary-value problems that had formerly been impossible, as we discuss in detail below. Hilbert's response went further. First off the mark in grasping the value of Fredholm's theory, he embarked on a deep investigation of integral equation methods, lecturing on them already in 1901-1902.

In 1898-99, after an extended period of research on the theory of numbers, Hilbert became interested in existence theory for PDE's. He announced that he had "saved" the Dirichlet principle for Laplace equation 1899, though this was not published until 1904. Here Hilbert found hypotheses sufficient to guarantee the existence of a solution to the variational problem associated with the Dirichlet problem for the Laplace equation, the solution having been assumed by Riemann (for example in connection with his proof of the Riemann mapping theorem). In his 1900 Paris lecture, Hilbert's problems 19 and 20 bear on related issues.

3.2. Hilbert's Problems and Existence Theory. Hilbert's famed 1900 Paris lecture included two problems having to do with partial differential equations, an index of the importance of the field. In the nineteenth problem, Hilbert noted that for some kinds of partial differential equations, for example the Laplace equation in two variables or the non-linear equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = e^y,$$

all the solutions are necessarily analytic functions. Here is how he puts the problem:

> 19. Are the solutions of regular problems in the calculus of variations always necessarily analytic? [Mentioning some examples, he continues] Most of these PDEs have the common characteristic of being the lagrangian differential equations of certain problems of variation, viz., of such problems of variation
>
> $$\int\int F(p, q, z; x, y)\,dxdy = minimum$$
>
> $$[p = z_x, q = z_y]$$
>
> as satisfy ... the inequality
>
> $$\frac{\partial^2 F}{\partial p^2} \cdots \frac{\partial^2 F}{\partial q^2} - \left(\frac{\partial^2 F}{\partial p \partial q}\right)^2 > 0,$$
>
> F itself being an analytic function. We shall call this sort of problem a *regular* variation problem...In other words *does every lagrangian partial differential equation of a regular variation problem have the property of admitting analytic integrals exclusively?* And is this the case when the function is constrained to assume, as, e.g., in Dirichlet's problem on the potential function, boundary values which are continuous but not analytic?

The twentieth problem, more generally, posed the question of existence of solutions to boundary value problems, which Hilbert felt would yield to a unified method based on the Dirichlet principle, provided that the notion of solution were to be suitably generalized. It should be noted that in a series of papers beginning in

1897, Hilbert rehabilitated the Dirichlet principle, freeing it from the Weierstrassian objections by clarifying the hypotheses necessary for the principle to be applied. Hilbert noted that, in the case of the Dirichlet problem for the Laplace equation, the existence issue was solved by the methods of C. Neumann, H. A. Schwarz, and Poincaré. Hilbert then continued:

> These methods, however, seem to be generally not capable of direct extension to the case where along the boundary there are prescribed either the differential coefficients or any relations between these and the values of the function. Nor can they be extended immediately to the case where the inquiry is not for potential surfaces but,say, for surfaces of least area, or surfaces of constant positive Gaussian curvature ... It is my conviction that it will be possible to prove these existence theorems by means of a general principle whose nature is indicated by the Dirichlet principle. This general principle will then perhaps enable us to approach the question: *Has not every regular variation problem a solution, provided certain assumptions regarding the given boundary conditions are satisfied... and provided also if need be that the notion of a solution shall be suitably extended?*

The nineteenth problem was solved rapidly in the affirmative by the Russian mathematician Serge Bernstein (1880-1968). Born in Odessa, where his father was a university anatomy lecturer, Bernstein went to Paris in 1898 where he began his studies in the École d'électrotechnique supérieure with an eye on an engineering career. He soon changed his direction and studied mathematics at the Faculty of Science at the Sorbonne, with a faculty including Poincaré and Picard. In 1902-1903 he spent the year in Göttingen, and it was during this period, with direct exposure to Hilbert, that he solved the nineteenth problem in the affirmative. He received a D. ès sci. maths from the Sorbonne for this work, in 1904, and then returned to Russia, where he had a long and distinguished career, first at Kharkov (1907-1933), then in Leningrad (1933-1943) and finally in Moscow. Rather oddly, his work on the twentieth problem was also thesis work, since foreign degrees were not recognized as a suitable credential for a university professorship at that time. Accordingly, Bernstein undertook studies, first for an M. Sc. at Kharkov. His main work on the twentieth problem is contained here, and it appeared between 1910 and 1912 in Russian.

Bernstein's suitability as a candidate to solve the nineteenth and twentieth problems rested on his deep familiarity with Picard's work. Bernstein had fully understood the 1890 method of successive approximations of Picard, as well as its later elaborations. He further shows a broad awareness of work in analysis, for example results published by Hurwitz which turned out to be key to his solution. I do not yet know what took him to Göttingen, but the visit was felicitous. In the 1904 paper presenting his solution to the 19th problem, he states that he was equally inspired by Picard and by Hilbert, the former for the methods, and the second for the goals. In particular, he said,

> I would like to thank very specially M. Hilbert for having personally suggested this interesting subject to me.

However astute Hilbert may have been as a judge of mathematical talent, we may note that he personally suggested this interesting subject to several other people at this time as well, an additional indication – beyond his published work in the area – that it was very much on his mind. Hilbert's students Charles A. Noble and E. R. Hedrick had succeeded in carrying out aspects of Hilbert's program on the twentieth problem shortly before. For the nineteenth, a 1902 dissertation by his student Lutkemeyer achieved a solution in a special case, with hypotheses which Bernstein was able to eliminate. This same result was achieved independently by Holmgren, the result appearing in the Math. Annalen of 1903.

In his 1904 paper, Bernstein's main theorem was the following:

> Let z be a function of x and y of which the first three derivatives are finite and continuous; it it is a solution of the equation
>
> $$F(x, y, z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}) = 0$$
>
> F being analytic, and if it satisfies the inequality
>
> $$4F_{z_{xx}} \cdot F_{z_{yy}} - \left(F_{z_{xy}}\right)^2 > 0$$
>
> then it is analytic.

Bernstein began his lengthy proof with a detailed analysis of the Picard method. He shows how to use it to prove the theorem of Lutkemeyer and Holmgren, deepening their understanding by eliminating an artificial hypothesis. He then analyzed its limitations, showing how it could not be used without modification for certain equations. His analysis of why it failed was decisive in his solution of the problem, and this will be discussed in the full version of the paper.

Bernstein's own research on the twentieth problem employed various strategies, frequently in combination. He employed ideas which were not widely appreciated until they were clarified and extended by Leray and Schauder in the 1930s. Bernstein assembled Poincaré's earlier idea of the continuation of solutions along a parameter (a homotopy type of argument) with that of obtaining a priori estimates on the partial derivatives in the equation. He was able to obtain existence results for a large class of second-order equations in two variables. The essential idea of the method is that for a solution, the partial derivatives in question can be bounded, where the bounds depend only on the structure of the equation, the boundary data, and the nature of the domain. The continuity method is then employed to deform the problem to a simpler one with known solution. For example, Bernstein was able by this means to prove the existence of solutions of the minimal surface equation

$$(1 + u_y)^2 u_{xx} - 2u_x u_y u_{xy} = (1 + u_x)^2 u_{yy} = 0,$$

for smooth boundary data and a convex region.

3.3. **Hilbert and Integral Equations.** Many of Hilbert's 50 Ph. D. students between 1898 and 1910 worked in the general area of partial differential equations and integral equations: Hedrick, Noble, Lutkemeyer, and others worked on classical aspects of the theory. Theses on integral equations began to appear in 1902, with the thesis of O. D. Kellogg. Max Mason in 1903,Erhard Schmidt in 1905, Hellinger, Weyl,Haar, etc. The work of Schmidt and Weyl in particular took a turn that gave prominence to a general viewpoint of the kind Leo Corry has referred to a "structural" in nature. Using the idea of a vector product of two functions in a function space, defined as an integral, Schmidt and others were able to provide a geometry for function spaces, leading ultimately to the structure now referred to as a Hilbert space.

Between 1902 and 1904 Hilbert realized that for integral equations of Fredholm type (Hilbert's type II, in the classification still used, with real symmetric kernel) there would be a theory analogous to that of orthogonal transformations of quadratic forms. He saw this as a unified viewpoint for "Schwingungslehre", as he termed linear problems in analysis, namely boundary value problems, and the associated study of eigenvalues and eigenfunction expansions for the corresponding operators. He rapidly moved from determinantal viewpoint to "function space" viewpoint, which is noticeable also in the work of his students, beginning with Schmidt. We also observe the usual Hilbertian emphasis on axiomatic theory .

Hilbert had looked at the question of the existence of "normal functions" in the theory of differential equations, and the development of arbitrary functions in terms of them, via the introduction of Green's functions. In his 1904 *Nachricht*, stimulated by Fredholm's work, he had reduced this (if reduced is the appropriate term) to the question of finding eigenfunctions corresponding to a symmetric kernel, and found the laws governing the development of arbitrary functions with respect to them.

A very rapid development between 1904-1910 led to a rather general viewpoint, maturing fully in 1920s following the work of Banach, though not to be fully exploited until the appropriate topological tools were developed in the 1940s and later. While we will discuss further some aspects of this research direction, our main aim here is to present it in contrast to the more classical turn. This is concentrated in the French and Italian contexts, and we turn to it now.

3.4. **Picard's Work on Integral Equations to 1906.** Returning to the French context, Picard saw Fredholm's *Comptes Rendus* articles, and probably also the Ph. D. thesis of Max Mason (at least a summary). Mason applied Hilbert's idea of using Fredholm's results to get existence of solutions for a large class of problems, e.g. the equation of vibrating plates, and the isoperimetric problem, by combining the Dirichlet principle idea with Fredholm's approach.

Picard, struck by the power of Fredholm's method, began lecturing on this by 1905-1906. However, Picard's original understanding, as we see from papers in 1904, appears to have emerged from a different context. He saw Fredholm's work at first in the functional equation context, relating it to earlier work of Volterra and his own student J. le Roux. In his first papers on the subject, [Picard 1904a] and [Picard 1904b], he saw this subject above all as a place where his own method of successive approximation could be put to good use, since once the problem is reformulated as an integral equation the method of successive approximations can be applied to the integral.

Two years later, in [Picard 1906a], he gave a fuller account via the *Rendiconti* of Palermo, for some time a favoured vehicle (like *Acta Mathematica*) for reaching a broad international audience with a longer paper. In this paper he positions Fredholm's work more in the context of Poincaré's efforts, (taking Fredholm at his word) namely as a method for existence and solution of problems in mathematical physics. In particular he considered linear equations of second order and the so-called biharmonic equation, a fourth-degree analogue of the Laplace equation. This biharmonic equation had been extensively studied by a number of Italian writers, as we have seen above, making the *Rendiconti* a natural venue for the paper on those grounds as well. The paper reformulates a number of such boundary-value problems as double or single layer problems, i.e. integral equations, then uses the Fredholm method to find eigenvalues and hence solution spaces.

It is clear that Hilbert's paper of 1904 has been read but only partly digested by Picard at this point. He was later able to come to a clearer understanding of Hilbert's work via the thesis of Schmidt, as we discuss below. Picard was already engaged in the PDE context, and with questions under discussion by Italian mathematicians such as T. Boggio and G. Lauricella. Given the novely of Hilbert's approach, it is thus hardly surprising that Picard does not work in the Hilbertian context at this point.

4. Italian Work and the Prix Vaillant

4.1. The Italian Tradition in Elasticity Theory.

4.2. The *Prix Vaillant*. A full discussion of the French and Italian reception should include an account of work on the vibration of elastic plates. This was posed by the *Académie des ciences* as the *Prix Vaillant* problem in 1906, and solved by Hadaamard, Lauricella, and Boggia, who shared the prize in 1907. Some of the details are discussed in [Maz'ya and Shaposhnikova 1998]

5. Further Research in the Period 1907-1915

5.1. The Lauricella-Picard Criterion of 1909. By 1909 Picard was to grasp Hilbert's work, in part through the intermediary of Erhard Schmidt. While Hilbert had restricted his investigation to the case of continuous kernels, Schmidt generalized this to square-integral kernels in his 1905 thesis. Schmidt likewise gave a geometric interpretation to Hilbert's results in his thesis, as Hilbert had not. Functions with singularities could thus serve as kernels, and in consequence of this Schmidt presented a spectral theory for non-symmetric operators. It was this theory that was used by Picard, as [Groetsch 2003] points out. In so doing, Picard clarified what is now seen as the essential difference between integral equations of first and second type, and gave an explicit criterion for the possibility of solving equations of type one. Picard developed this theory in 1909, and presented it in his lectures of that year as well as giving a short account in the *Comptes rendus* of 14 June 1909 [Picard 1909a]. The same result was found independently by Lauricella. No priority dispute resulted, and the issue of independent discovery seems clear, the more likely since the question is natural and the result not difficult at the level of generality employed by Picard and Lauricella.

In this research Picard and Lauricella demonstrate their mastery of essential results due to Erhard Schmidt and Friedrich Riesz. Schmidt's result, from his dissertation of July 1905, had been published in the *Mathematische Annalen* in 1907 (and also in CR??). Schmidt's result states the following. Suppose we are given two "conjugate" integral equations

$$\phi(x) = \lambda \int_a^b K(x,y)\psi(y)dy \tag{5.1}$$

$$\psi(x) = \lambda \int_a^b K(y,x)\phi(y)dy \tag{5.2}$$

and suppose that, in Hilbert's terminology, $\lambda_1, \lambda_2, \ldots$ are the eigenvalues of the problem, that is the values of λ that correspond to non-trivial solutions ϕ_i and ψ_i. Then the ϕs form an orthonormal system, as we would now say (Picard says orthogonal and normal, following Schmidt). Picard combined this with a version of Riesz's result, the theorem we now know as the Riesz-Fischer theorem, likewise from 1907. In Picard's statement, this says the following: a kernel function $f(x)$ can be found such that a given sequence a_n can correspond to the "Fourier" coefficients with respect to a given orthonormal collection of functions if and only if the sequence is square-summable, that is, $\sum a_n^2$ is convergent.

The system $(\phi_n, \psi_n, \lambda_n)$ is what would now be called a *singular system* of the kernel K. To understand what Picard had, it is useful to state a modern version of the theorem, as it is given in the classic work [Smithies 1965], with some changes in notation.

Theorem 1. *Let $(\phi_n, \psi_n, \lambda_n)$ be a singular system of the $\mathcal{L}^2$ kernel $K(s,t)$ and let $y(s)$ be a given $\mathcal{L}^2$ function. Then the equation*

$$y(s) \approx \int K(s,t)x(t)dt$$

has an $\mathcal{L}^2$ solution $x(t)$ if and only if (a)

$$\sum_{n=1}^{\infty} \lambda_n^2 |(y, \phi_n)|^2 < +\infty,$$

and (b) $(y, \phi) = 0$ for every L2 function ϕ such that $K^*\phi \approx 0$.

The $\approx$ here indicates equality except on a set of measure 0, and the $(,)$ refers to the usual operation of projection onto the basis functions. [3]

Picard assumes in addition that the functions ϕ_i form what he, again following Schmidt, terms a closed sequence (*fermé*), which means that there is no function $h(x)$ other than the zero function such that $\int_a^b h(x)\phi_n(x)dx = 0$. Nowadays we would note that this means the sequence is complete as a basis for the entire function space, that is, the null space of the operator adjoint to the one defined by the integral equation is trivial. Under these circumstances, Picard was able to show that a necessary and sufficient condition that an integral equation of type 1 be solved is the criterion that

$$\sum \lambda_n^2 a_n^2$$

is convergent, where a_n are the components of the kernel with respect to the given basis. These are referred to as Fourier coefficients by Picard, following Riesz. The proof is straightforward if one uses the ideas of Schmidt and Riesz (and Hilbert) and thus this demonstrates nicely that by 1909 Picard had indeed grasped these ideas fully and was able to put them to good use.[4] It will be convenient to give both directions of the proof of Picard's criterion.

[3]See [Tricomi 1957] on this result, pp. 88-90 on Riesz-Fischer 1907 and 143-150 on this criterion in the symmetric and non-symmetric cases. Tricomi casts all this in terms of convergence in the mean. In that case, the so-called Weyl lemma, the analogue of the Cauchy convergence criterion for convergence in the mean, is required.The Weyl lemma is from a 1909 paper by Weyl, Math Ann 67, 1909, 225-245. Tricomi observes that the Riesz-Fischer result is "One of the first brilliant successes of the concept of the Lebesgue integral", p. 88. This is because the function f corresponding to a sequence a_i of Fourier coefficients with respect to the system ϕ_i with $\sum a_i^2 < \infty$ need not be Riemann integrable. Weyl was aware of this early on, as we see in his thesis p. 16 footnote. Lebesgue's *Leçons* appeared in 1904.

[4]Picard certainly steers clear of issues involving sets of measure 0 and so on, however, by his hypotheses. Thus the theorem is a bit simpler than in more recent statements such as that of Smithies, p. 164.

The given integral equation is

$$f(x) = \int_a^b K(x,y)F(y)dy \tag{5.3}$$

where K and f are known and F is to be found. In [Picard 1909a] Picard supposes that the functions f and K are continuous, though as he observes it would be possible to weaken this condition.[5] Let F be a solution, and let a_n be the Fourier coefficients (in his terminology) of f with respect to the sequence ϕ_n. Then

$$a_n = \int_a^b f(x)\phi_n(x)dx = \int_a^b \int_a^b K(x,y)\phi_n(y)dxdy$$

and from 5.1 we have at once

$$a_n = \frac{1}{\lambda_n}\int_a^b F(y)\psi_n(y)dy.$$

But the integral here gives the coeffients B_n of F with respect to ψ_n, so $B_n = \lambda_n a_n$. But by the Riesz-Fischer theorem, these are square-summable, so the convergence of

$$\sum \lambda_n^2 a_n^2$$

is necessary for the solution to exist.

Conversely, suppose the Picard criterion holds for f. Then, again by Riesz-Fischer, there is a function F which has the Fourier coefficients $\lambda_n a_n$, using the same notation. We now show that a solution to 5.3 is given by this F.

First, let

$$f_1(x) = \int_a^b K(x,y)F(y)dy,$$

so that the Fourier coefficients of f_1 are given as

$$\int_a^b f_x(x)\phi_n(x)dx = \int_z^b \int_a^b K(x,y)\phi_n(x)F(y)dxdy = \frac{1}{\lambda_n}\int_a^b F(y)\psi_n(y)dy = a_n.$$

Picard now uses his hypothesis that there is no function $h(x)$ other than the zero function such that $\int_a^b h(x)\phi_n(x)dx = 0$. An immediate consequence is that $f_1 = f$, which means that F is a solution of 5.3. [6]

6. Textbooks and Mongraphs 1909-1915

The rapid development of the theory of integral equations, and its immediate important application to the solution of boundary-value problems, led to its presentation in lectures in Germany, France, Italy (England?) and the USA. Between 1909 and 1915 Maximc Bôcher, Adolph Kneser, Trajan Lalesco, the duo of H. B. Heywood and M. Fréchet, Édouard Goursat, Vio Volterra and Robert Adhémar all gave textbook accounts, most of them based on lectures. Some of these treatments were incorporated in longer works on analysis (for example, Goursat's, which nonetheless runs to 220 pages) while others were free-standing accounts.

[5] Beginning with [Picard 1909b], he assumed that, as in Riesz's paper, F is integrable and square-integrable, that is, an $mathcalL^2$ function.

[6] One may wonder at this point whether the connection between abgeschlossenheit and vollständigkeit is clear to any of these people.

6.1. **Bôcher.** Maxime Bôcher (1867-1918) studied at Harvard, then went to Göttingen where he completed a Ph. D. in 1891. He arrived at a point in time when Felix Klein was interested in potential theory, and his dissertation concerns series expansions of potential functions (i.e. functions satisfying Laplace's equation), employing the then-popular *cyclides* as a device to unify various kinds of special functions which are employed depending on the geometry of the problem. He was unable to solve the problem of convergence of these expansions, which remained open for many years, and his interest in integral equations may have arisen partly from the fact the Hilbert's methods offered the possibility of progress in that direction. [7]

Bôcher completed his *Introduction to the Study of Integral Equations* in November of 1908. It appeared in the series *Cambridge Tracts in Mathematics and Mathematical Physics* edited by G. H. Hardy and E. Cunningham, and had a second edition in 1914. This seventy-page account Bôcher's account pushes the subject back to Fourier, following Hermann Weyl in considering Fourier an unconscious user of integral equations. Bôcher looks at the historical roots of integral functional equations in the work of Abel and Dirichlet, and his account devotes quite a bit of attention to historical developments. Bôcher argues that the prospective importance of the theory was already noted by Du Bois-Reymond (Crelle 103, 1888, 228) or even earlier by Rouché (CR 51 1860 126). As for what the importance consists of, Bôcher is clear:

> ...like so many other branches of analysis the theory was called into being by specific problems in mechanics and mathematical physics. Thes was true not merely in the early days of Abel and Liouville, but also more recently in the cases of Volterra and Fredholm. Such applications or the theory, together with its relations to other branches of analysis, are what give the subject its great importance. [Bôcher 1909], 2.

He is not specific about the issue of "relations to other branches of analysis", stating in a footnote: "cf., for instance, much of Hilbert's work.".

The book is a very thorough elementary presentation, starting with work by Abel and Liouville, continuing with Volterra and Fredholm, and the work of Hilbert and Schmidt on equations with symmetric kernel. Particular attention is paid to giving a rigorous account of Fredholm's work. There is nothing original as far as the results are concerned, and most of the description is based on the work of others, notable Fredholm, Hilbert, and Volterra. No account is given at all of the application to boundary-value problems, though there is some mention of the relationship to series expansions, notably the 1907 work of Kneser.

Despite the importance of applications, he expressly excludes them from his account, which means that this book has at most an indirect effect on research in PDEs. Though it is useful as an introduction to the subject in English, it appeared a bit prematurely in the development of the theory.

6.2. **Lalesco and Heywood & Fréchet.** . Trajan Lalesco (1882-1929) was a student of Picard, completing a doctoral thesis in Paris on integral equations of

[7]At any rate, Osgood, in an obituary of Bôcher, mentions the fact that Hilbert's work is relevant here. [Osgood 1919], 238).

the first kind in 1908. [8] The aim of Lalesco's book is to give an account of the theory (and not the applications) to that date. Accordingly, he treats the Volterra and Fredholm equations, discusses the work of Hilbert on symmetric kernels and Schmidt's extension of that theory. The corresponding eigenfunction expansions are treated in some detail. The concluding portion of the book gives a preliminary discussion of singular integral equations, following the work of Hermann Weyl; however, it is not a complete account of that theory.

In contrast, the book by Heywood and Fréchet restricts itself to the Fredholm equation, and specifically aims at giving an account of applications to mathematical physics. This book comes from the orbit of Hadamard, rather than Picard. Fréchet was very much a protégé of Hadamard, (Taylor, Maz'ya and Shaposhnikova), and Heywood was his doctoral student (I think). The theory here is fully attached to the applications. In fact, the first chapter sets up a considerable number of boundary-value problems as integral equations, beginning with an introduction to potential theory and the idea of a Green's function. Once the appropriate methods are developed, the final chapter of the book concerns the solution of these problems. The procedure is done in a very step-by-step fashion, and indicates the author's faith in the notion that the Fredholm approach is one which will be generally used by anyone interested in the solution of such equations, notably physicists.

7. Concluding Remarks

The distinct approaches belonging to different national schools were becoming merged inthe textbook tradition by 1914, and as we observed French and Italian writers were beginning to be more cognizant of Hilbert's methods and approaches by that time. The story was then complicated by the advent of World War I. It is difficult now to appreciate th extent to which this disrupted scientific communication internationally. The mailing of German journals to Britain and France, later to Italy and even Japan, was suspended and communication in the other direction soon ceased as well. Differential equations belonged to a sensitive area, and in particular we know of cases in the British context where material relevant to aeronautics ceased to be published. Each side decried the vicious barbarism of the other; in France, divorces could be granted if one spouse called the other a *boche*. Picard lost a son, and this man who had begun his scientific career as one involved in the creation of an international community became a bitter enemy of German re-admission to the ranks of international scientific councils. German scientists were excluded from international mathematical meetings until Bologna in 1928, and there was no official delegation until 1932. All this led to a complex relationship between the various European mathematical communities, one which acted to hinder international cultivation of research programs in many areas. It is against this background that subsequent developments must be understood.

8. Appendix 1: the Fredholm Theorem

We now consider why these series correspond to the determinant of a linear system. In the integral equation 2.1, ψ and f are real-valued functions on $[0,1]$

[8] His name is more properly taken as Traian Lalescu, as it is written in Rumanian. The French spelling was used in most of the early literature, however.

and $[0,1] \times [0,1]$ respectively. While Fredholm doubtless was thinking of partitions, suppose instead that ψ and f are step functions, where

$$f(x,y) = f_{rs} \quad \left(\frac{r-1}{n} < x < \frac{r}{n}, \frac{s-1}{n} < x < \frac{s}{n}\right)$$
$$\psi(x) = \psi_r \quad \left(\frac{r-1}{n} < x < \frac{r}{n}\right).$$

In this case, the equation 2.1 becomes

$$\phi(x) = \psi_r - \sum_{s=1}^{n} f_{rs} \int_{\frac{s-1}{n}}^{\frac{s}{n}} \phi(y)dy \quad x \in \left[\frac{r-1}{n}, \frac{r}{n}\right]. \tag{8.1}$$

This implies in turn that ϕ must be a step function, so that

$$\phi_r = \psi_r - \frac{1}{n}\sum_{s=1}^{n} \phi_r,$$

so the analogy to a system of linear equations is clear. An expression involving the kernel function f takes the place of the coefficients, and if, as Fredholm did, we follow this analogy, we seek a determinant involving the f values to which we can apply Cramer's rule. (Where are the eigenvalues??). Changing the sign of the sum to reflect modern usage, the determinant of this system is

$$\begin{vmatrix} 1-\frac{1}{n}f_{11} & 0-\frac{1}{n}f_{12} & \dots & 0-\frac{1}{n}f_{1n} \\ 0-\frac{1}{n}f_{21} & 1-\frac{1}{n}f_{22} & \dots & 0-\frac{1}{n}f_{2n} \\ & & \dots & \\ 0-\frac{1}{n}f_{n1} & 0-\frac{1}{n}f_{n2} & \dots & 1-\frac{1}{n}f_{nn} \end{vmatrix}. \tag{8.2}$$

To get a finite expansion for this determinant, we may now repeatedly apply a standard determinantal identity for determinants with a row which is the sum of two addends. In the 2×2 case this reads

$$\begin{vmatrix} x_{11}+y_{11} & x_{12}+y_{12} \\ x_{21} & x_{22} \end{vmatrix} = \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} + \begin{vmatrix} y_{11} & y_{12} \\ x_{21} & x_{22} \end{vmatrix}.$$

Using this in every row leads to the following expansion of (8.2):

$$1 - \frac{1}{n}S_1 + \left(\frac{1}{n}\right)^2 S_2 + \dots + (-1)^n \left(\frac{1}{n}\right)^n S_n.$$

Here S_m is the sum of all the *principal* minors of $\det(f_{rs})$, where by a principal minor we mean one where, having chosen any m row numbers, we choose the m columns with the same index and arrange the rows in increasing order of index. (Another way of saying this is to note that the diagonal elements all have the same row and column subscript.) This expansion would have been reasonably well known to Fredholm's contemporaries, since it was a standard tool in the theory of quadratic forms.[9]

[9]The result is apparently due to Cauchy [Cauchy 1815]. It is described in detail in various textbook presentations, for example by Brioschi, who points out an application related to celestial mechanics. See [Brioschi 1854],47-48, [Borchardt 1847], 54.

Select Bibliography

[Bôcher 1909] Bôcher, Maxim. *An Introduction to the Theory of Integral Equations.* Cambridge, Cambridge University Press, 1909.

[Borchardt 1847] Borchardt, C. W. "Développements sur l'équation àl'aide de laquelle on détermine les inégalités séculaires du mouvement des plantètes.", *Journal des mathématiques pures et appliquées*, 12, 1847.

[Brioschi 1854] Brioschi, Francesco, *La Teorica dei Determinanti e le sue Principali Applicazioni.* Pavia, n.p., 1854.

[Cauchy 1815] "Mémoire sur les fonctions qui ne peuvent obtenir que deux valeurs égales et de signes contraires par suite des transpositions opérées sur les caractères qu'elles renferment." *Journal de l'École polytechnique* Cahier XVII, t. 10, p.92?. Also in *Œuvres complètes d'Augustin Cauchy*, sér. 2, T. 1, 92-169. The paper was delivered at the *Institut* on Nov. 30, 1812.

[Dieudonné 1981] Dieudonné, Jean. *History of Functional Analysis.* Amsterdam, North Holland, 1981.

[Fréchet 1906] Fréchet, Maurice. "Sur quelques points du calcul fonctionnel." *Rendiconti del Circolo Matematico de Palermo* 22, 1906, 1-74. This was his thesis at Paris.

[Fredholm 1899] Fredholm, Ivar. "Sur une classe d'équations aux dérivées partielles". *Comptes rendus de l'Academie des Sciences*, 129, 1899, 32-34.

[Fredholm 1900] Fredholm, Ivar. "Title here". *Stockholm Öfver.* 57, 39-46, 1900.

[Fredholm 1902] Fredholm, Ivar. "Sur une classe de transformations rationelles". *Comptes rendus de l'Academie des Sciences*, 134, 219-222, 1902.

[Fredholm 1903] Fredholm, Ivar. "Sur une classe d'équations fonctionelles", *Acta Math.*, 27, pages, 1903.

[Groetsch 2003] Groetsch, Charles W. "The Delayed Emergence of Regularization Theory," *Boll. Storia Sci. Mat.* XXIII, 105-120, 2003.

[Goursat 1915] Goursat, E. *Traité d'analyse.* Second Edition, Paris 1915. On Integral equations see v. 3, chaps 30-33, 324-544.

[Hellinger 1935] Hellinger, Ernst. "Hilberts Arbeiten über Integralgleichungen und unendliche Gleichungssysteme". in [Hilbert Ges. Abh.] III, 95-145.

[Hellinger and Toeplitz 1927] Hellinger, Ernst, and Otto Toeplitz. "Integralgleichungen und Gleichungen mit unendlichviele unbekannten", *Enzyklopädie der mathematischen Wissenschaften* II.3.2, section IIC. 13 1927, 1335-1597.

[Heywood and Fréchet 1912] Heywood, H. BB. and M. Fréchet. *L'équation de Fredholm et ses application á la physique mathématique.* Préface de J. Hadamard. Paris, Hermann, 1912.

[Hilbert 1900] "Sur le principe de Dirichlet." Trans. M. Laugel. *Nouvelles Annales de Mathématiques* 3sér. 19, 337-344, 1900.

[Hilbert 1904] Hilbert, David. "Über das Dirichlet'sche Prinzip," *Math. Ann.* 59 (1904), 161-186. In [Hilbert Ges. Abh.], v. 3, 15.37.

[Hilbert Ges. Abh.] Hilbert, David. *Gesammelte Abhandlungen.* Berlin, Springer, 1935. Cited here from the Second edition, 1970.

[Lalesco 1912] *Introduction à la théorie des é quation intégrales.* Préface de E. Picard. Paris, Hermann, 1912.

[Le Roux 1895] Le Roux, J. "Sur les intégrales des équations linéaires". *Ann. Sci. ENS*(III)12, 1895, 227-316.

[Le Roux 1903] Le Roux, J. "Recherches sur les équations aux dérivées partielles." *Journal de mathématiques pures et appliquées*, 5 sér. 9, 403-455, 1903.

[Lützen 1990] Joseph Liouville 1809 – 1882: Master of Pure and Applied Mathematics. Springer, New York, 1990.

[Mason 1903] Mason, Charles Max *Randweertaufgaben bei gewöhnlichen Differentialgleichungen.* Ph. D. thesis, Göttingen, 1903.

[Maz'ya and Shaposhnikova 1998] Maz'ya, V. and Shaposhnikova, T. *Jacques Hadamard, a Universal Mathematician*, Providence: American Mathematical Society, 1998.

[Nabonnand 1999] Nabonnand, Philippe, ed. *La correspondance entre Henri Poincaré et Gösta Mittag-Leffler.* Basel, Birkhäuser, 1999.

[Osgood 1919] Osgood, William Fogg. "The Life and Services of Maxime Bôcher." *Bull. American Math. Soc.* 25, 1919, 337-350.

[Pedersen 1988] Pedersen, Gert K. *Analysis Now.* Springer, New York, 1988.

[Picard 1981] Picard, Émile. *Œuvres de Charles-Émile Picard.* Paris, CNRS, 1981. 4 volumes.

[Picard 1904a] Picard, Émile. "Sur certaines solutions doublement périodiques de quelques équations aux dérivées partielles". *Comptes rendus de l'Academie des Sciences* Séance du 25 janvier 1904. T. 138, 1904, 181-183. In [Picard 1981], 2, 687-689.

[Picard 1904b] Picard, Émile. "Sur une équation fonctionelle". *Comptes rendus de l'Academie des Sciences* Séance du 25 juillet 1904. T. 139, 1904, 245-248. In [Picard 1981], 4, 313-316.

[Picard 1906a] Picard, Émile. "Sur quelques problèmes de Physique mathématique se rattachant à l'équation de M. Fredholm". *Comptes rendus de l'Academie des Sciences* Séance du 9 avril 1906. T. 142, 861-865. In [Picard 1981], 4, 317-321.

[Picard 1906b] Picard, Émile. "Sur quelques applications de l'équation foncionelle de M. Fredholm". *Rendiconti de Palermo* 22, 241-259. In [Picard 1981], 4, 323-341.

[Picard 1909a] Picard, Émile. "Quelques remarques sur les équations intégrales de première espèce et sur certains problèmes de Physique mathématique." *Comptes rendus de l'Academie des Sciences* Séance du 14 juin 1909. T. 148, 1563-1568. In [Picard 1981], 4, 379-384.

[Picard 1909b] Picard, Émile. "Sur les équations intégrales de première espèce." *Comptes rendus de l'Academie des Sciences* Séance du 28 juin 1909. T. 148, 1707-1708. In [Picard 1981], 4, 385-386.

[Picard 1910] Picard, Émile. "Sur un théorème général relatif aux équations intégrales de première espèce et sur quelques problèmes de physique mathématique". *Rendiconti Circ. Mat. Palermo* 29, 1910, 79-97.

[Plancherel 1912] Plancherel, M. "La théorie des équations intégrales. *Enseignement mathématique*14, 1912, 89-107.

[Poincaré Œuvres] Poincaré, Henri. *Œuvres.* Date; place.

[Poincaré 1890] Poincaré, H. "Sur les équations aux dérivées partielles de la physique mathématique," *American Journal of Mathematics,* 12(1890), 211-294. *Oeuvres,* tome IX, 28-113.

[Poincaré 1894] Poincaré, H. "Sur les équations de la physique mathématique, *Rendiconti del Circolo matematico di Palermo,* 8(1894), 57-155. *Oeuvres,* tome IX, 123-196.

[Poincaré 1897] Poincaré, Henri. "La méthode de Neumann et le probl'eme de Dirichlet."*Acta Math* 20, 1897, 59-142. Dated Nov. 11, 1895.

[Poincaré 1908a] Poincaré, Henri."Remarques sur l'équation de Fredholm".*Comptes rendus de l'Academie des Sciences* 147, 1908, 1367-1371.In [Poincaré Œuvres] 3,540-544.

[Poincaré 1908b] Poincaré, Henri."Sur quelques applications de la méthode de Fredholm".*Comptes rendus de l'Academie des Sciences* 148, 1908, 125.126.In [Poincaré Œuvres] 3,545-546.

[Smithies 1965] Smithies, F. *Integral Equations.* Cambridge, Cambridge University Press, 1965.

[Tricomi 1957] *Integral Equations.* Interscience, New York, 1957. Page references here from the 1985 reprint by Dover, New York.

[Truesdell and Toupin 1960] C.A. Truesdell and R.A. Toupin, 1960. The classical field theories, in: Handbuch der Physik (S. Flügge, ed., Berlin) III, pp. 226-790.

[Zeilon 1930] Zeilon, Nils. "Biographie de Ivar Fredholm". *Acta Mathematica,* 54, 1930, I-XVI.

E-mail address: `rossana.Tazzioli@math.univ-lille1.fr,tarchi@sfu.ca`

Université de Lille I

Department of Mathematics, Simon Fraser University