THE MICROSCOPICALLY-DESCRIPTIVE FLUID EQUATIONS BY BOLTZMANN

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ABSTRACT. The microscopically-description of hydromechanics equations are followed by the description of equations of gas theory by Maxwell, Kirchhoff and Boltzmann. Above all, in 1872, Boltzmann formulated the Boltzmann equations, expressed by the following today's formulation :

$$\partial_t f + \mathbf{v}\cdot\nabla_{\mathbf{x}} f = Q(f,g), \quad t>0, \quad \mathbf{x},\mathbf{v}\in\mathbb{R}^n (n\geq 3), \quad \mathbf{x}=(x,y,z), \quad \mathbf{v}=(\xi,\eta,\zeta), \tag{1}$$

$$Q(f,g)(t,x,v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,\sigma)\{g(v'_*)f(v') - g(v_*)f(v)\}d\sigma dv_*, \quad g(v'_*) = g(t,x,v'_*), \text{ etc.} \tag{2}$$

These equations are able to be reduced for the general form of the hydrodynamic equations, after the formulations by Maxwell and Kirchhoff, and from which the Euler equations and the Navier-Stokes equations are reduced as the special case.

After Stokes' linear equations, the equations of gas theories were deduced by Maxwell in 1865, Kirchhoff in 1868 and Boltzmann in 1872, They contributed to formulate the fluid equations and to fix the Navier-Stokes equations, when Prandtl stated the today's formulation in using the nomenclature as the "so-called Navier-Stokes equations" in 1934, in which Prandtl included the three terms of nonlinear and two linear terms with the ratio of two coefficients as 3 : 1, which arose Poisson in 1831, Saint-Venant in 1843, and Stokes in 1845.

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CONTENTS

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1. Introduction

[1]

We have studied the original microscopically descriptive Navier-Stokes ($MDNS$) equations as the progenitors [2], Navier, Cauchy, Poisson, Saint-Venant and Stokes, and endeavor to ascertain their aims and conceptual thoughts in formulations these new equation. "The two-constant theory" was introduced first introduced in 1805 by Laplace [3] in regard to capillary action with constants denoted by H and K.

Thereafter, various pairs of constants have been proposed by their progenitors in formulating $MDNS$ equations or equations describing equilibrium or capillary situations. It is commonly accepted that this theory describes isotropic, linear elasticity. [4] We can find the "two-constant" in the equations of gas theories by Maxwell, Kirchhoff and Boltzmann, which were fixed into the common linear terms, and which originally takes its rise in Poisson and Stokes.

The gas theorists studied also the general equations of hydromechanics, which have the same proportion of coefficients as the equations deduced by Poisson and Stokes with only the linear term and the ratio of the coefficient of Laplacian to that of gradient of divergence term is 3 : 1. (cf Table 2.)

2. A universal method for the two-constant theory

In this section, we propose a universal method to describe the kinetic equations that arise in isotropic, linear elasticity. This method is outlined as follows:

- The partial differential equations describing waves in elastic solids or flows in elastic fluids are expressed by using one constant or a pair of constants C_1 and C_2 such that:

$$\text{for elastic solids:} \qquad \frac{\partial^2 \mathbf{u}}{\partial t^2} - (C_1 T_1 + C_2 T_2) = \mathbf{f},$$

$$\text{for elastic fluids:} \qquad \frac{\partial \mathbf{u}}{\partial t} - (C_1 T_1 + C_2 T_2) + \cdots = \mathbf{f},$$

  where $T_1,\ T_2, \cdots$ are the terms depending on tensor quantities constituting our equations. For example, the NS equations corresponding to incompressible fluids consist of the kinetic equation along with the continuity equation and are conventionally written, in modern vector notation, as follows:

$$\frac{\partial \mathbf{u}}{\partial t} - \mu \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f}, \quad \text{div } \mathbf{u} = 0. \tag{3}$$

  Here $\mathbf{u}$ is the velocity, $\mathbf{f}$ accounts for the body forces present, p the pressure and $\Delta \equiv \nabla \cdot \nabla$ the Laplacian operator.

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[1] $(\Downarrow)$ Throughout this paper, in citation of bibliographical sources, we show our own paragraph or sentences of commentaries by surrounding between $(\Downarrow)$ and $(\Uparrow)$. $((\Uparrow)$ is used only when not following to next section,). And by $=^*$, we detail the statement by original authors, because we would like to discriminate and to avoid confusion from the descriptions by original authors. The mark : $\Rightarrow$ means transformation of the statements in brevity by ours. And all the frames surrounding the statements are inserted for important remark of ours. Of course, when the descriptions are explicitly distinct without these marks, these are not the descriptions in citation of bibliographical sources.

[2] $(\Downarrow)$ To establish a time line of these contributor, we list for easy reference the year of their birth and death: Sir I.Newton(1643-1727), D.Bernoulli(1700-1782), Euler(1707-1783), d'Alembert(1717-1783), Lagrange(1736-1813), Laplace(1749-1827), Fourier(1768-1830), Gauss(1777-1855), Navier(1785-1836), Poisson(1781-1840), Cauchy(1789-1857), Saint-Venant(1797-1886), Stokes(1819-1903). The order in our paper below is by date of proposal or publication.

[3] $(\Downarrow)$ Of capillary action, Laplace[21, V.4, Supplement p.2] achnowledges Clailaut [6, p.22], and Clailaut cites Maupertuis.

[4] $(\Downarrow)$ Darrigol [10, p.121].

- The two coefficients C_1 and C_2 associated with the tensor terms are the two constants of the theory, definitions of which depend on the contributing author. For example, ε and E were introduced by Navier, R and G by Cauchy, k and K in elastic and $(K+k)\alpha$ and $\frac{(K+k)\alpha}{3}$ in fluid by Poisson, ε and $\frac{\varepsilon}{3}$ by Saint-Venant, and μ and $\frac{\mu}{3}$ by Stokes. Since Poisson, the ratio of two coefficient in fluid was fixed at 3. Moreover, C_1 and C_2 can be expressed in the following form:

$$\begin{cases} C_1 \equiv \mathcal{L} r_1 g_1 S_1, \\ C_2 \equiv \mathcal{L} r_2 g_2 S_2, \end{cases} \quad \begin{cases} S_1 = \iint g_3 \to C_3, \\ S_2 = \iint g_4 \to C_4, \end{cases} \quad \Rightarrow \quad \begin{cases} C_1 = C_3 \mathcal{L} r_1 g_1 = \frac{2\pi}{15} \mathcal{L} r_1 g_1, \\ C_2 = C_4 \mathcal{L} r_2 g_2 = \frac{2\pi}{3} \mathcal{L} r_2 g_2. \end{cases}$$

  Here $\mathcal{L}$ corresponds to either $\sum_0^\infty$ as argued for by Poisson or $\int_0^\infty$ as argued for by Navier. A heated debate had developed between the two over this point. It is a matter of personnel preference as to how the two constants should be expressed.
- The two constants depend on two radial functions r_1 and r_2 related to the radius of the active sphere of the molecules, raised to some power of n for Poisson's and Navier's cases; the relationship between these functions can be expressing by a logarithm with base r such that: $\log_r \frac{r_1}{r_2} = 2$.
- g_1 and g_2 are the kernel functions having both
  - the physical caracteristics come from the fluid dynamics described by the microscopically basic relations of the attraction and/or repulsion and
  - the mathematical requirements for the rapidly decreasing function.
- S_1 and S_2 are two expressions which determine the angular dependence on the surface of the active unit-sphere centered on a molecule through application of the double integral (or single sum in the case of Poisson's fluid).
- g_3 and g_4 are certain compound spherical harmonic functions determining the momentum over the unit sphere.
- C_3 and C_4 are indirectly determined as the common coefficients derived from the invariant tensor. With the exception of Poisson's fluid case, C_3 of C_1 is $\frac{2\pi}{3}$, and C_4 of C_2 is $\frac{2\pi}{15}$, which are evaluated over the unit spheres for each molecule, and which are independent of the preference in using integrals or summations. In Poisson's case, we obtain the same values as the above after multiplying by $\frac{1}{4\pi}$. The integrals are calculated from the total momentum of the active sphere surrounding the molecule.
- The ratio of C_3 to C_4 : $\frac{C_3}{C_4} = \frac{1}{5}$ including Poisson's case.

2.1. Poisson's Fluid pressure in motion.

- § 63.

[5] Poisson's tensor of the pressures in fluid reads as follows :

$(7\text{-}7)_{Pf}$

$$\begin{bmatrix} U_1 & U_2 & U_3 \\ V_1 & V_2 & V_3 \\ W_1 & W_2 & W_3 \end{bmatrix} = \begin{bmatrix} \beta\left(\frac{du}{dz} + \frac{dw}{dx}\right) & \beta\left(\frac{du}{dy} + \frac{dv}{dx}\right) & p - \alpha\frac{d\psi t}{dt} - \frac{\beta'}{\chi t}\frac{d\chi t}{dt} + 2\beta\frac{du}{dx} \\ \beta\left(\frac{dv}{dz} + \frac{dw}{dy}\right) & p - \alpha\frac{d\psi t}{dt} - \frac{\beta'}{\chi t}\frac{d\chi t}{dt} + 2\beta\frac{dv}{dy} & \beta\left(\frac{du}{dy} + \frac{dv}{dx}\right) \\ p - \alpha\frac{d\psi t}{dt} - \frac{\beta'}{\chi t}\frac{d\chi t}{dt} + 2\beta\frac{dw}{dz} & \beta\left(\frac{dv}{dz} + \frac{dw}{dy}\right) & \beta\left(\frac{du}{dz} + \frac{dw}{dx}\right) \end{bmatrix},$$

$$(k+K)\alpha = \beta, \quad (k-K)\alpha = \beta', \quad p = \psi t = K, \quad \text{then} \quad \beta + \beta' = 2k\alpha, \tag{4}$$

where χt is the density of the fluid around the point M, and ψt is the pressure. Here we can replace the first column with the third one, then we see easily the conventional style of array as follows :

$$\begin{bmatrix} U_3 & U_2 & U_1 \\ V_3 & V_2 & V_1 \\ W_3 & W_2 & W_1 \end{bmatrix} = \begin{bmatrix} p - \alpha\frac{d\psi t}{dt} - \frac{\beta'}{\chi t}\frac{d\chi t}{dt} + 2\beta\frac{du}{dx} & \beta\left(\frac{du}{dy} + \frac{dv}{dx}\right) & \beta\left(\frac{du}{dz} + \frac{dw}{dx}\right) \\ \beta\left(\frac{du}{dy} + \frac{dv}{dx}\right) & p - \alpha\frac{d\psi t}{dt} - \frac{\beta'}{\chi t}\frac{d\chi t}{dt} + 2\beta\frac{dv}{dy} & \beta\left(\frac{dv}{dz} + \frac{dw}{dy}\right) \\ \beta\left(\frac{du}{dz} + \frac{dw}{dx}\right) & \beta\left(\frac{dv}{dz} + \frac{dw}{dy}\right) & p - \alpha\frac{d\psi t}{dt} - \frac{\beta'}{\chi t}\frac{d\chi t}{dt} + 2\beta\frac{dw}{dz} \end{bmatrix},$$

[5](⇓) In Poisson [35], the title of the chaper 7 is "*Calcul des Pressions dans les Fluides en mouvement ; équations defferentielles de ce mouvement.*"

TABLE 1. The two constants in the kinetic equations

no	name	problem	C_1	C_2	C_3	C_4	$\mathcal{L}$	r_1	r_2	g_1	g_2	remark
1	Navier [31]	elastic solid	ε		$\frac{2\pi}{15}$		$\int_0^\infty d\rho$	ρ^4		$f\rho$		ρ : radius
2	Navier [32]	fluid	ε		$\frac{2\pi}{15}$		$\int_0^\infty d\rho$	ρ^4		$f(\rho)$		ρ : radius
				E		$\frac{2\pi}{3}$	$\int_0^\infty d\rho$		ρ^2		$F(\rho)$	
3	Cauchy [5]	system of particles in elastic and fluid	R		$\frac{2\pi}{15}\Delta$		$\int_0^\infty dr$	r^3		$f(r)$		$f(r) \equiv \pm[r\mathrm{f}'(r) - \mathrm{f}(r)]$
				G		$\frac{2\pi}{3}\Delta$	$\int_0^\infty dr$		r^3		$\pm\mathrm{f}(r)$	$f(r) \neq \mathrm{f}(r)$, $\Delta = \frac{\mathcal{M}}{\mathcal{V}}$: mass of molecules per volume.
4	Poisson [34]	elastic solid	k		$\frac{2\pi}{15}$		$\sum \frac{1}{\alpha^5}$	r^5		$\frac{d.\frac{1}{r}fr}{dr}$		
				K		$\frac{2\pi}{3}$	$\sum \frac{1}{\alpha^5}$		r^3		fr	
5	Poisson [35]	elastic solid and fluid	k		$\frac{1}{30}$		$\sum \frac{1}{\varepsilon^3}$	r^3		$\frac{d.\frac{1}{r}fr}{dr}$		$C_3 = \frac{1}{4\pi}\frac{2\pi}{15} = \frac{1}{30}$
				K		$\frac{1}{6}$	$\sum \frac{1}{\varepsilon^3}$		r		fr	$C_4 = \frac{1}{4\pi}\frac{2\pi}{3} = \frac{1}{6}$
6	Saint-Venant [41]	fluid	ϵ	$\frac{\epsilon}{3}$								
7	Stokes [42]	fluid	μ	$\frac{\mu}{3}$								
8	Stokes [42]	elastic solid	A	B								$A = 5B$

The elements of velocity $\mathbf{u} = (u, v, w)$ are :

$$\frac{dx}{dt} = u, \quad \frac{dy}{dt} = v, \quad \frac{dz}{dt} = w$$

$$\begin{cases} \frac{d^2x}{dt^2} = \frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}, \\ \frac{d^2y}{dt^2} = \frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}, \\ \frac{d^2z}{dt^2} = \frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz} \end{cases}$$

$$\varpi \equiv p - \alpha\frac{d\psi t}{dt} - \frac{\beta + \beta'}{\chi t}\frac{d\chi t}{dt}, \tag{5}$$

[a]

$$(7\text{-}9)_{Pf} \qquad \begin{cases} \rho(X - \frac{d^2x}{dt^2}) = \frac{d\varpi}{dx} + \beta(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}), \\ \rho(Y - \frac{d^2y}{dt^2}) = \frac{d\varpi}{dy} + \beta(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}), \\ \rho(Z - \frac{d^2z}{dt^2}) = \frac{d\varpi}{dz} + \beta(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}). \end{cases} \tag{6}$$

[a](⇓) $(7\text{-}9)_{Pf}$ means the equation number with chapter of Poisson [35]

If we put $\mathbf{f} = (X, Y, Z)$ then (6) becomes as follows :

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\beta}{\rho}\Delta\mathbf{u} + \frac{1}{\rho}\nabla\varpi = \mathbf{f} \tag{7}$$

2.2. Stokes' comment on Poisson's fluid equations.

Stokes comments on Poisson's $(7\text{-}9)_{Pf}$ as follows :

TABLE 2. The kinetic equations of the hydrodynamics until the "Navier-Stokes equations" was fixed. (Rem. HD : hydro-dynamics, N under entry-no : non-linear, gr.dv : grad.div, E : $\frac{\Delta}{gr.dv}$ of elastic, F : $\frac{\Delta}{gr.dv}$ of fluid)

no	name/prob	the kinetic equations	Δ	gr.dv	E	F
1 N	Euler (1752-55) [?, p.127] fluid	$\begin{cases} X - \frac{1}{h}\frac{dp}{dx} = \frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}, \\ Y - \frac{1}{h}\frac{dp}{dy} = \frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}, \\ Z - \frac{1}{h}\frac{dp}{dz} = \frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}, \end{cases} \quad \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0.$				
2	Navier (1827)[31] elastic solid	$(6\text{-}1)_{Ne} \quad \begin{cases} \frac{\Pi}{g}\frac{d^2x}{dt^2} = \varepsilon\left(3\frac{d^2x}{da^2} + \frac{d^2x}{db^2} + \frac{d^2x}{dc^2} + 2\frac{d^2y}{dbda} + 2\frac{d^2z}{dcda}\right), \\ \frac{\Pi}{g}\frac{d^2y}{dt^2} = \varepsilon\left(\frac{d^2y}{da^2} + 3\frac{d^2y}{db^2} + \frac{d^2y}{dc^2} + 2\frac{d^2x}{dadb} + 2\frac{d^2z}{dcdb}\right), \\ \frac{\Pi}{g}\frac{d^2z}{dt^2} = \varepsilon\left(\frac{d^2z}{da^2} + \frac{d^2z}{db^2} + 3\frac{d^2z}{dc^2} + 2\frac{d^2x}{dadc} + 2\frac{d^2y}{dbdc}\right) \end{cases}$ where Π is density of the solid, g is acceleration of gravity.	ε	2ε	$\frac{1}{2}$	
3 N	Navier (1827)[32] fluid	$\begin{cases} \frac{1}{\rho}\frac{dp}{dx} = X + \varepsilon\left(3\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + 2\frac{d^2v}{dxdy} + 2\frac{d^2w}{dxdz}\right) - \frac{du}{dt} - \frac{du}{dx}\cdot u - \frac{du}{dy}\cdot v - \frac{du}{dz}\cdot w\ ; \\ \frac{1}{\rho}\frac{dp}{dy} = Y + \varepsilon\left(\frac{d^2v}{dx^2} + 3\frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} + 2\frac{d^2u}{dxdy} + 2\frac{d^2w}{dydz}\right) - \frac{dv}{dt} - \frac{dv}{dx}\cdot u - \frac{dv}{dy}\cdot v - \frac{dv}{dz}\cdot w\ ; \\ \frac{1}{\rho}\frac{dp}{dz} = Z + \varepsilon\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + 3\frac{d^2w}{dz^2} + 2\frac{d^2u}{dxdz} + 2\frac{d^2v}{dydz}\right) - \frac{dw}{dt} - \frac{dw}{dx}\cdot u - \frac{dw}{dy}\cdot v - \frac{dw}{dz}\cdot w\ ; \end{cases}$	ε	2ε		$\frac{1}{2}$
4	Cauchy (1828)[5] system of particles in elastic and fluid	$\begin{cases} (L+G)\frac{\partial^2\xi}{\partial x^2} + (R+H)\frac{\partial^2\xi}{\partial y^2} + (Q+I)\frac{\partial^2\xi}{\partial z^2} + 2R\frac{\partial^2\eta}{\partial x\partial y} + 2Q\frac{\partial^2\zeta}{\partial z\partial x} + X = \frac{\partial^2\xi}{\partial t^2}, \\ (R+G)\frac{\partial^2\eta}{\partial x^2} + (M+H)\frac{\partial^2\eta}{\partial y^2} + (P+I)\frac{\partial^2\eta}{\partial z^2} + 2P\frac{\partial^2\zeta}{\partial y\partial z} + 2R\frac{\partial^2\xi}{\partial x\partial y} + Y = \frac{\partial^2\eta}{\partial t^2}, \\ (Q+G)\frac{\partial^2\zeta}{\partial x^2} + (P+H)\frac{\partial^2\zeta}{\partial y^2} + (N+I)\frac{\partial^2\zeta}{\partial z^2} + 2Q\frac{\partial^2\xi}{\partial z\partial x} + 2P\frac{\partial^2\eta}{\partial y\partial z} + Z = \frac{\partial^2\zeta}{\partial t^2}, \end{cases}$ $G = H = I, \quad L = M = N, \quad P = Q = R, \quad L = 3R$	$R+G$	$2R$	if $G = 0$ $\frac{1}{2}$	
5	Poisson (1831)[35] elastic solid in general equations	$\begin{cases} X - \frac{d^2u}{dt^2} + a^2\left(\frac{d^2u}{dx^2} + \frac{2}{3}\frac{d^2v}{dydx} + \frac{2}{3}\frac{d^2w}{dzdx} + \frac{1}{3}\frac{d^2u}{dy^2} + \frac{1}{3}\frac{d^2u}{dz^2}\right) = \frac{\Pi}{\rho}\frac{d^2u}{dx^2}, \\ Y - \frac{d^2v}{dt^2} + a^2\left(\frac{d^2v}{dy^2} + \frac{2}{3}\frac{d^2u}{dxdy} + \frac{2}{3}\frac{d^2w}{dzdy} + \frac{1}{3}\frac{d^2v}{dx^2} + \frac{1}{3}\frac{d^2v}{dz^2}\right) = \frac{\Pi}{\rho}\frac{d^2v}{dy^2}, \\ Z - \frac{d^2w}{dt^2} + a^2\left(\frac{d^2w}{dz^2} + \frac{2}{3}\frac{d^2u}{dxdz} + \frac{2}{3}\frac{d^2v}{dydz} + \frac{1}{3}\frac{d^2w}{dx^2} + \frac{1}{3}\frac{d^2w}{dy^2}\right) = \frac{\Pi}{\rho}\frac{d^2w}{dz^2}, \end{cases}$	$\frac{a^2}{3}$	$\frac{2a^2}{3}$	$\frac{1}{2}$	
6	Poisson (1831)[35] fluid in general equations	$\begin{cases} \rho(\frac{Du}{Dt} - X) + \frac{dp}{dx} + \alpha(K+k)\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) + \frac{\alpha}{3}(K+k)\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \\ \rho(\frac{Dv}{Dt} - Y) + \frac{dp}{dy} + \alpha(K+k)\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right) + \frac{\alpha}{3}(K+k)\frac{d}{dy}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \\ \rho(\frac{Dw}{Dt} - Z) + \frac{dp}{dz} + \alpha(K+k)\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\right) + \frac{\alpha}{3}(K+k)\frac{d}{dz}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \end{cases}$ $\begin{cases} \rho(X - \frac{d^2x}{dt^2}) = \frac{d\varpi}{dx} + \beta(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}), \\ \rho(Y - \frac{d^2y}{dt^2}) = \frac{d\varpi}{dy} + \beta(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}), \\ \rho(Z - \frac{d^2z}{dt^2}) = \frac{d\varpi}{dz} + \beta(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}) \end{cases}$ WHERE $\varpi \equiv p - \alpha\frac{d\psi t}{dt} - \frac{\beta+\beta'}{\chi t}\frac{d\chi t}{dt}, \quad \beta \equiv -\alpha(K+k)$	β	$\frac{\beta}{3}$		3
7	Saint-Venant (1843)[41] fluid	His equations are none in [41], however his tensor is in Table 5 (4).	ε	$\frac{\varepsilon}{3}$		3
8	Stokes (1849)[42] fluid	$(12)_S \quad \begin{cases} \rho(\frac{Du}{Dt} - X) + \frac{dp}{dx} - \mu\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) - \frac{\mu}{3}\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \\ \rho(\frac{Dv}{Dt} - Y) + \frac{dp}{dy} - \mu\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right) - \frac{\mu}{3}\frac{d}{dy}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \\ \rho(\frac{Dw}{Dt} - Z) + \frac{dp}{dz} - \mu\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\right) - \frac{\mu}{3}\frac{d}{dz}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0. \end{cases}$	μ	$\frac{\mu}{3}$		3
9	Maxwell (1865-66) [29] HD	$\begin{cases} \rho\frac{\partial u}{\partial t} + \frac{dp}{dx} - C_M\left[\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + \frac{1}{3}\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)\right] = \rho X, \\ \rho\frac{\partial v}{\partial t} + \frac{dp}{dy} - C_M\left[\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} + \frac{1}{3}\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)\right] = \rho Y, \\ \rho\frac{\partial w}{\partial t} + \frac{dp}{dz} - C_M\left[\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2} + \frac{1}{3}\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)\right] = \rho Z \end{cases}$ where, $C_M \equiv \frac{pM}{6k\rho\Theta_2}$	C_M	$\frac{C}{3}$		3
10	Kirchhoff (1876)[18] HD	$\begin{cases} \mu\frac{du}{dt} + \frac{\partial}{\partial x} - C_K\left[\Delta u + \frac{1}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = \mu X, \\ \mu\frac{dv}{dt} + \frac{\partial p}{\partial y} - C_K\left[\Delta v + \frac{1}{3}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = \mu Y, \\ \mu\frac{dw}{dt} + \frac{\partial p}{\partial z} - C_K\left[\Delta z + \frac{1}{3}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = \mu Z, \end{cases} \begin{cases} \frac{1}{\mu}\frac{d\mu}{dt} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \\ \text{where, } C_K \equiv \frac{1}{3\kappa}\frac{p}{\mu} \end{cases}$	C_K	$\frac{\Delta}{3}$		3
11 N	Rayleigh (1883)[40] HD	$\begin{cases} \frac{1}{\rho}\frac{dp}{dx} = -\frac{du}{dt} + \nu\nabla^2 u - u\frac{du}{dx} - v\frac{du}{dy}, \\ \frac{1}{\rho}\frac{dp}{dy} = -\frac{dv}{dt} + \nu\nabla^2 v - u\frac{dv}{dx} - v\frac{dv}{dy} \end{cases}, \quad \frac{du}{dx} + \frac{dv}{dy} = 0$	ν			
12	Boltzmann (1895)[1] HD	$(221)_B \quad \begin{cases} \rho\frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} - \mathcal{R}\left[\Delta u + \frac{1}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = \rho X, \\ \rho\frac{\partial v}{\partial t} + \frac{\partial p}{\partial y} - \mathcal{R}\left[\Delta v + \frac{1}{3}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = \rho Y, \\ \rho\frac{\partial w}{\partial t} + \frac{\partial p}{\partial z} - \mathcal{R}\left[\Delta w + \frac{1}{3}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = \rho Z \end{cases}$	$\mathcal{R}$	$\frac{\mathcal{R}}{3}$		3
13 N	Prandtl (1905)[38] HD	$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) + \nabla(V+p) = k\nabla^2 v, \quad \text{DIV } \mathbf{v} = 0$	k			
14 N	Prandtl (1934)[39] HD	$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\nu}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right),$ FOR INCOMPRESSIBLE, IT IS SIMPLIFIED DIV $\mathbf{w} = 0, \quad \frac{D\mathbf{w}}{dt} = g - \frac{1}{\rho}\text{GRAD } p + \nu\Delta\mathbf{w}$	ν	$\frac{\nu}{3}$		3

TABLE 3. Geneology of tensors

1	name	tensor (3×3)	coefficient matrix (3 × 5) in equations
1-1	Navier elasticity	$t_{ij} = -\varepsilon(\delta_{ij}u_{k,k} + u_{i,j} + u_{j,i})$ (5-4)$_{Ne}$ $-\varepsilon\begin{bmatrix} \left(3\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right) & \left(\frac{du}{dy}+\frac{dv}{dx}\right) & \left(\frac{dw}{dx}+\frac{du}{dz}\right) \\ \left(\frac{du}{dy}+\frac{dv}{dx}\right) & \left(\frac{du}{dx}+3\frac{dv}{dy}+\frac{dw}{dz}\right) & \left(\frac{dv}{dz}+\frac{dw}{dy}\right) \\ \left(\frac{dw}{dx}+\frac{du}{dz}\right) & \left(\frac{dv}{dz}+\frac{dw}{dy}\right) & \left(\frac{du}{dx}+\frac{dv}{dy}+3\frac{dw}{dz}\right) \end{bmatrix}$ $= -\varepsilon\begin{bmatrix} \epsilon+2\frac{du}{dx} & \frac{du}{dy}+\frac{dv}{dx} & \frac{dw}{dx}+\frac{du}{dz} \\ \frac{du}{dy}+\frac{dv}{dx} & \epsilon+2\frac{dv}{dy} & \frac{dv}{dz}+\frac{dw}{dy} \\ \frac{dw}{dx}+\frac{du}{dz} & \frac{dv}{dz}+\frac{dw}{dy} & \epsilon+2\frac{dw}{dz} \end{bmatrix}$, where $\epsilon = \frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}$	We define the coefficient matrix in elasticity : C^e_T : the coefficient of $\begin{bmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial y^2} & \frac{\partial^2 u}{\partial z^2} & \frac{\partial^2 v}{\partial x\partial y} & \frac{\partial^2 w}{\partial z\partial x} \\ \frac{\partial^2 v}{\partial x^2} & \frac{\partial^2 v}{\partial y^2} & \frac{\partial^2 v}{\partial z^2} & \frac{\partial^2 w}{\partial y\partial z} & \frac{\partial^2 u}{\partial x\partial y} \\ \frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial y^2} & \frac{\partial^2 w}{\partial z^2} & \frac{\partial^2 u}{\partial z\partial x} & \frac{\partial^2 v}{\partial y\partial z} \end{bmatrix}$, then (6-1)$_{Ne}$ $\Rightarrow$ $C^e_T = -\varepsilon\begin{bmatrix} 3&1&1&2&2 \\ 1&3&1&2&2 \\ 1&1&3&2&2 \end{bmatrix}$
1-2	Navier fluid	$t_{ij} = (p-\varepsilon u_{k,k})\delta_{ij} - \varepsilon(u_{i,j}+u_{j,i})$ $\begin{bmatrix} \epsilon' - 2\varepsilon\frac{du}{dx} & -\varepsilon(\frac{du}{dy}+\frac{dv}{dx}) & -\varepsilon(\frac{dw}{dx}+\frac{du}{dz}) \\ -\varepsilon(\frac{du}{dy}+\frac{dv}{dx}) & \epsilon'-2\varepsilon\frac{dv}{dy} & -\varepsilon(\frac{dv}{dz}+\frac{dw}{dy}) \\ -\varepsilon(\frac{dw}{dx}+\frac{du}{dz}) & -\varepsilon(\frac{dv}{dz}+\frac{dw}{dy}) & \epsilon'-2\varepsilon\frac{dw}{dz} \end{bmatrix}$, where $\epsilon' = p - \varepsilon(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz})$	Similarly, we define the coefficient matrix in fluid : C^f_T, which contains p in (1,1)-, (2,2)- and (3,3)-element. $C^f_T = \begin{bmatrix} p-3\varepsilon & -\varepsilon & -\varepsilon & -2\varepsilon & -2\varepsilon \\ -\varepsilon & p-3\varepsilon & -\varepsilon & -2\varepsilon & -2\varepsilon \\ -\varepsilon & -\varepsilon & p-3\varepsilon & -2\varepsilon & -2\varepsilon \end{bmatrix}$
2	Cauchy system (contains both elasticity and fluid)	$t_{ij} = \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j}+v_{j,i})$ (60)$_C$ $\begin{bmatrix} k\frac{\partial\xi}{\partial a}+K\nu & \frac{k}{2}\left(\frac{\partial\xi}{\partial b}+\frac{\partial\eta}{\partial a}\right) & \frac{k}{2}\left(\frac{\partial\zeta}{\partial a}+\frac{\partial\xi}{\partial c}\right) \\ \frac{k}{2}\left(\frac{\partial\xi}{\partial b}+\frac{\partial\eta}{\partial a}\right) & k\frac{\partial\eta}{\partial b}+K\nu & \frac{k}{2}\left(\frac{\partial\eta}{\partial c}+\frac{\partial\zeta}{\partial b}\right) \\ \frac{k}{2}\left(\frac{\partial\zeta}{\partial a}+\frac{\partial\xi}{\partial c}\right) & \frac{k}{2}\left(\frac{\partial\eta}{\partial c}+\frac{\partial\zeta}{\partial b}\right) & k\frac{\partial\zeta}{\partial c}+K\nu \end{bmatrix}$, where $\nu = \frac{\partial\xi}{\partial a}+\frac{\partial\eta}{\partial b}+\frac{\partial\zeta}{\partial c}$	(46)$_C \Rightarrow C^e_T = \begin{bmatrix} L&R&Q&2R&2Q \\ R&M&P&2P&2R \\ Q&P&N&2Q&2P \end{bmatrix}$ $\Rightarrow R\begin{bmatrix} 3&1&1&2&2 \\ 1&3&1&2&2 \\ 1&1&3&2&2 \end{bmatrix}$, where $P=Q=R,\quad L=M=N,\quad L=3R.$
3-1	Poisson elasticity	$t_{ij} = -\frac{a^2}{3}(\delta_{ij}u_{k,k}+u_{i,j}+u_{j,i})$ (6)$_{Pe}$ $-\frac{a^2}{3}\begin{bmatrix} \epsilon+2\frac{du}{dx} & \frac{du}{dy}+\frac{dv}{dx} & \frac{dw}{dx}+\frac{du}{dz} \\ \frac{du}{dy}+\frac{dv}{dx} & \epsilon+2\frac{dv}{dy} & \frac{dv}{dz}+\frac{dw}{dy} \\ \frac{dw}{dx}+\frac{du}{dz} & \frac{dv}{dz}+\frac{dw}{dy} & \epsilon+2\frac{dw}{dz} \end{bmatrix}$, where $\epsilon = \frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}$	(6)$_{Pe}$ $\begin{cases} X - \frac{d^2u}{dt^2} \\ +a^2\left(\frac{d^2u}{dx^2}+\frac{2}{3}\frac{d^2v}{dydx}+\frac{2}{3}\frac{d^2w}{dzdx}+\frac{1}{3}\frac{d^2u}{dy^2}+\frac{1}{3}\frac{d^2u}{dz^2}\right)=0, \\ Y - \frac{d^2v}{dt^2} \\ +a^2\left(\frac{d^2v}{dy^2}+\frac{2}{3}\frac{d^2u}{dxdy}+\frac{2}{3}\frac{d^2w}{dzdy}+\frac{1}{3}\frac{d^2v}{dx^2}+\frac{1}{3}\frac{d^2v}{dz^2}\right)=0, \\ Z - \frac{d^2w}{dt^2} \\ +a^2\left(\frac{d^2w}{dz^2}+\frac{2}{3}\frac{d^2u}{dxdz}+\frac{2}{3}\frac{d^2v}{dydz}+\frac{1}{3}\frac{d^2w}{dx^2}+\frac{1}{3}\frac{d^2w}{dy^2}\right)=0, \end{cases}$ $\Rightarrow C^e_T = -\frac{a^2}{3}\begin{bmatrix} 3&1&1&2&2 \\ 1&3&1&2&2 \\ 1&1&3&2&2 \end{bmatrix}$
3-2	Poisson fluid	$t_{ij} = -p\delta_{ij} + \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j}+v_{j,i})$ (7-7)$_{Pf}$ $\begin{bmatrix} \beta\left(\frac{du}{dz}+\frac{dw}{dx}\right) & \beta\left(\frac{du}{dy}+\frac{dv}{dx}\right) & \pi+2\beta\frac{du}{dx} \\ \beta\left(\frac{dv}{dz}+\frac{dw}{dy}\right) & \pi+2\beta\frac{dv}{dy} & \beta\left(\frac{du}{dy}+\frac{dv}{dx}\right) \\ \pi+2\beta\frac{dw}{dz} & \beta\left(\frac{dv}{dz}+\frac{dw}{dy}\right) & \beta\left(\frac{du}{dz}+\frac{dw}{dx}\right) \end{bmatrix}$, where $\pi = p - \alpha\frac{d\psi t}{dt} - \frac{\beta'}{\chi t}\frac{d\chi t}{dt}$	(7-9)$_{Pf}$ $\Rightarrow$ $C^f_T = \begin{bmatrix} \varpi+\beta & \beta & \beta & 0 & 0 \\ \beta & \varpi+\beta & \beta & 0 & 0 \\ \beta & \beta & \varpi+\beta & 0 & 0 \end{bmatrix}$. According to Stokes: if we put $\varpi = p + \frac{\alpha}{3}(K+k)\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)$ $\Rightarrow$ $C^f_T = \begin{bmatrix} p+\frac{4\beta}{3} & \beta & \beta & \frac{\beta}{3} & \frac{\beta}{3} \\ \beta & p+\frac{4\beta}{3} & \beta & \frac{\beta}{3} & \frac{\beta}{3} \\ \beta & \beta & p+\frac{4\beta}{3} & \frac{\beta}{3} & \frac{\beta}{3} \end{bmatrix}$ $\Rightarrow (12)_S$. Remark: $\alpha(K+k) = \beta$.
4	Saint-Venant fluid	$t_{ij} = (\frac{1}{3}(P_{xx}+P_{yy}+P_{zz}) - \frac{2\varepsilon}{3}v_{k,k})\delta_{ij} + \varepsilon(v_{i,j}+v_{j,i})$ $= (-p - \frac{2\varepsilon}{3}v_{k,k})\delta_{ij} + \varepsilon(v_{i,j}+v_{j,i})$ $\begin{bmatrix} \pi+2\varepsilon\frac{d\xi}{dx}, & \varepsilon\left(\frac{d\xi}{dy}+\frac{d\eta}{dx}\right) & \varepsilon\left(\frac{d\zeta}{dx}+\frac{d\xi}{dz}\right) \\ \varepsilon\left(\frac{d\xi}{dy}+\frac{d\eta}{dx}\right) & \pi+2\varepsilon\frac{d\eta}{dy} & \varepsilon\left(\frac{d\eta}{dz}+\frac{d\zeta}{dy}\right) \\ \varepsilon\left(\frac{d\zeta}{dx}+\frac{d\xi}{dz}\right) & \varepsilon\left(\frac{d\eta}{dz}+\frac{d\zeta}{dy}\right) & \pi+2\varepsilon\frac{d\zeta}{dz} \end{bmatrix}$, where $\pi = \frac{1}{3}\left(P_{xx}+P_{yy}+P_{zz}\right) - \frac{2\varepsilon}{3}\left(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\right)$ $\equiv -p - \frac{2\varepsilon}{3}\left(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\right)$	no description in [41].
5	Stokes fluid	$t_{ij} = (-p - \frac{2}{3}\mu v_{k,k})\delta_{ij} + \mu(v_{i,j}+v_{j,i})$, tensor = −1 × $\begin{bmatrix} p-2\mu\left(\frac{du}{dx}-\delta\right) & -\mu\left(\frac{du}{dy}+\frac{dv}{dx}\right) & -\mu\left(\frac{dw}{dx}+\frac{du}{dz}\right) \\ -\mu\left(\frac{du}{dy}+\frac{dv}{dx}\right) & p-2\mu\left(\frac{dv}{dy}-\delta\right) & -\mu\left(\frac{dv}{dz}+\frac{dw}{dy}\right) \\ -\mu\left(\frac{dw}{dx}+\frac{du}{dz}\right) & -\mu\left(\frac{dv}{dz}+\frac{dw}{dy}\right) & p-2\mu\left(\frac{dw}{dz}-\delta\right) \end{bmatrix}$ where $3\delta = \frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}$	(12)$_S$ $\Rightarrow$ $C^f_T = \begin{bmatrix} -p+\frac{4\mu}{3} & \mu & \mu & \frac{\mu}{3} & \frac{\mu}{3} \\ \mu & -p+\frac{4\mu}{3} & \mu & \frac{\mu}{3} & \frac{\mu}{3} \\ \mu & \mu & -p+\frac{4\mu}{3} & \frac{\mu}{3} & \frac{\mu}{3} \end{bmatrix}$. Remark: $\frac{4}{3}\mu = 2\mu(1-\frac{1}{3})$

TABLE 4. Geneology of tensors (continued.)

1	name	tensor (3×3)
6	Maxwell fluid	$t_{ij} = (-p - \frac{2}{3}\mu v_{k,k})\delta_{ij} + \mu(v_{i,j} + v_{j,i})$, $\begin{bmatrix} p - \frac{M}{9k\rho\Theta_2}p\left(2\frac{du}{dx} - \frac{dv}{dy} - \frac{dw}{dz}\right) & -\frac{M}{6k\rho\Theta_2}p\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & -\frac{M}{6k\rho\Theta_2}p\left(\frac{dw}{dx} + \frac{du}{dz}\right) \\ -\frac{M}{6k\rho\Theta_2}p\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & p - \frac{M}{9k\rho\Theta_2}p\left(\frac{du}{dx} - 2\frac{dv}{dy} - \frac{dw}{dz}\right) & -\frac{M}{6k\rho\Theta_2}p\left(\frac{dv}{dz} + \frac{dw}{dy}\right) \\ -\frac{M}{6k\rho\Theta_2}p\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & -\frac{M}{6k\rho\Theta_2}p\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & p - \frac{M}{9k\rho\Theta_2}p\left(\frac{du}{dx} - \frac{dv}{dy} - 2\frac{dw}{dz}\right) \end{bmatrix}$
7	Kirchhoff fluid	$t_{ij} = (-p - 2kv_{l,l})\delta_{ij} + k(v_{i,j} + v_{j,i})$, $\begin{bmatrix} p - 2k\frac{\partial u}{\partial x} & -k\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & -k\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\ -k\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & p - 2k\frac{\partial v}{\partial y} & -k\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ -k\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & -k\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & p - 2k\frac{\partial w}{\partial z} \end{bmatrix}$
8	Boltzmann fluid	$t_{ij} = (-p - \frac{2}{3}\mu v_{k,k})\delta_{ij} + \mu(v_{i,j} + v_{j,i})$, $\begin{bmatrix} p - 2\mathcal{R}\left\{\frac{\partial u}{\partial x} - \frac{1}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right\} & -\mathcal{R}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & -\mathcal{R}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\ -\mathcal{R}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & p - 2\mathcal{R}\left\{\frac{\partial v}{\partial y} - \frac{1}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right\} & -\mathcal{R}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ -\mathcal{R}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & -\mathcal{R}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & p - 2\mathcal{R}\left\{\frac{\partial w}{\partial z} - \frac{1}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right\} \end{bmatrix}$ where, $\mathcal{R} = \frac{M}{6k\rho\Theta_2}p$.

On this supposition we shall get the value of $\frac{d\Psi t}{dt}$ from that of $R_1' - K$ in the equations of page 140 by putting

$$\frac{du}{dx} = \frac{dv}{dy} = \frac{dw}{dz} = -\frac{1}{3\chi t}\frac{d\chi t}{dt}.$$

We have therefore

$$\alpha\frac{d\chi t}{dt} = \frac{\alpha}{3}(K - 5k)\frac{d\chi t}{\chi t dt}.$$

$$(7\text{-}2)_{Pf} \qquad \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = -\frac{1}{\chi t}\frac{d\chi t}{dt}. \tag{8}$$

Putting now for $\beta + \beta'$ its value $2\alpha k$, and for $\frac{1}{\chi t}\frac{d\chi t}{dt}$ its value given by equation (8) [6], the expression for ϖ, page 152, [7] becomes

$$\varpi = p - \alpha\frac{d\psi t}{dt} - \frac{\beta + \beta'}{\chi t}\frac{d\chi t}{dt} = p - \left(\frac{\alpha}{3}(K - 5k) + 2\alpha k\right)\frac{d\chi t}{\chi t dt} = p + \frac{\alpha}{3}(K + k)\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right).$$

Observing that $\alpha(K + k) = \beta$, this value of ϖ reduces Poisson's equation $(7\text{-}9)_{Pf}$ [=(6)] to the equation $(12)_S$ of this paper. ([42, p.119]).

Namely, by using $\alpha(K + k) = \beta$ in (4), we get the following :

$$\begin{cases} \frac{d\varpi}{dx} = \frac{dp}{dx} + \frac{\beta}{3}\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right), \\ \frac{d\varpi}{dy} = \frac{dp}{dy} + \frac{\beta}{3}\frac{d}{dy}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right), \\ \frac{d\varpi}{dz} = \frac{dp}{dz} + \frac{\beta}{3}\frac{d}{dz}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right), \end{cases}$$

then (6)($= (7\text{-}9)_{Pf}$) turns out :

[6](⇓) Poisson[35, p.141]
[7](⇓) cf. (5)

$$\begin{cases} \rho(\frac{Du}{Dt}-X)+\frac{dp}{dx}+\alpha(K+k)\Big(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\Big)+\frac{\alpha}{3}(K+k)\frac{d}{dx}\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big)=0,\\ \rho(\frac{Dv}{Dt}-Y)+\frac{dp}{dy}+\alpha(K+k)\Big(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}\Big)+\frac{\alpha}{3}(K+k)\frac{d}{dy}\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big)=0,\\ \rho(\frac{Dw}{Dt}-Z)+\frac{dp}{dz}+\alpha(K+k)\Big(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+\frac{d^2w}{dz^2}\Big)+\frac{\alpha}{3}(K+k)\frac{d}{dz}\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big)=0, \end{cases}$$

$$\Rightarrow (12)_S \quad \begin{cases} \rho(\frac{Du}{Dt}-X)+\frac{dp}{dx}-\mu\Big(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\Big)-\frac{\mu}{3}\frac{d}{dx}\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big)=0,\\ \rho(\frac{Dv}{Dt}-Y)+\frac{dp}{dy}-\mu\Big(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}\Big)-\frac{\mu}{3}\frac{d}{dy}\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big)=0,\\ \rho(\frac{Dw}{Dt}-Z)+\frac{dp}{dz}-\mu\Big(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+\frac{d^2w}{dz^2}\Big)-\frac{\mu}{3}\frac{d}{dz}\Big(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\Big)=0. \end{cases}$$

Therefore, Poisson contains both compressible and incompressible fluid.

3. Genealogy and settlement of the stress tensor

In Figure 1, we have traced the genealogy of the tensor terms, in particular noting the form of each tensor t_{ij} appearing in the NS equations. These tensors are listed in Table 5, where we have differentiated those tensors associated with elastic solids or elastic fluids. From this genealogy, it could be asserted that Cauchy [4, 5] was the first user of "tensors" and arguably its inventor. This view is supported by the admission of Poisson [35] that he received the idea of a "symmetric tensor" from Cauchy. Moreover, the idea of tensor by Saint-Venant concurs with the work of Stokes. Here, we denote the two routes as NCP and PSS, both of which are portrayed in our figure, and by which we can explain the genealogy of tensor as it applies to the NS equations. cf. Table 5.

Fig.1: A genealogy of the stress tensors in the prototypical Navier-Stokes equations

Navier[31, 32]⊗
↗ ‖ ⇕ ↘
(Euler)⇒ ‖ Poisson[34, 35]• ⇒ Saint-Venant[41]† ⇒ Stokes[42]‡ ⇒ Maxwell[29]♭ ⇒ Kirchhoff♯ ⇒ Boltzmann♮
↘ ⇓ ⇑⊙ ↗⇒
Cauchy[4, 5]§
← MD deduction → ‖ ← Non-MD deduction → ← Gas theory by MD deduction
← NCP pattern → ‖ ← PSS pattern →

Legend for superscripted marks:

⊗ Navier:	$t^e_{ij}=-\varepsilon(\delta_{ij}u_{k,k}+u_{i,j}+u_{j,i}),\ t^f_{ij}=(p-\varepsilon u_{k,k})\delta_{ij}-\varepsilon(u_{i,j}+u_{j,i})$
• Poisson:	$t^e_{ij}=-\frac{a^2}{3}(\delta_{ij}u_{k,k}+u_{i,j}+u_{j,i}),\ t^f_{ij}=-p\delta_{ij}+\lambda v_{k,k}\delta_{ij}+\mu(v_{i,j}+v_{j,i})$
§ Cauchy:	$t^{e,f}_{ij}=\lambda v_{k,k}\delta_{ij}+\mu(v_{i,j}+v_{j,i})$
† Saint-Venant:	$t^f_{ij}=(\frac{1}{3}(P_{xx}+P_{yy}+P_{zz})-\frac{2\varepsilon}{3}v_{k,k})\delta_{ij}+\varepsilon(v_{i,j}+v_{j,i}),\ \frac{1}{3}(P_{xx}+P_{yy}+P_{zz})=-p$
‡ Stokes:	$t^f_{ij}=(-p-\frac{2}{3}\mu v_{k,k})\delta_{ij}+\mu(v_{i,j}+v_{j,i}),$
⊙ Poisson:	stated his reduction of the number of independent t_{ij} from 9 to 6 is due to Cauchy. (cf.§??)
♭ Maxwell:	$t_{ij}=(-p-\frac{2}{3}\mu v_{k,k})\delta_{ij}+\mu(v_{i,j}+v_{j,i})$
♯ Kirchhoff[18]:	$t^f_{ij}=(-p-2kv_{l,l})\delta_{ij}+k(v_{i,j}+v_{j,i})$
♮ Boltzmann[1]:	$t^f_{ij}=(-p-\frac{2}{3}\mathcal{R}v_{k,k})\delta_{ij}+\mathcal{R}(v_{i,j}+v_{j,i})$

We cannot ascribe to Euler a definite form for the stress tensor; however, Voigt[45] has presented a version in 1905. [8] He begins by introducing an exterior subscript index of the vector as also interior

[8](⇓) As an aside, W.Voigt [45] states Euler equations with his invented tensor in 1905 as follows : (we show his sketched contents)

> Auch hier sind die Ausdrücke für die Componenten nach den Richtungen der Tensoren T_1, T_2, T_3 - auf denen eine Seite hervorzuheben ist - von Interesse ; es gilt nämlich, wenn diese Richtungen wieder durch die Indices 1, 2, 3 characterisirt werden, höchst einfach

$$(19)_V \qquad [\mathcal{B}.T]_1=\mathcal{B}_1.T_1,\cdots$$

indices to the product of elements.

$$[\mathcal{B}.T]_1 = \mathcal{B}_1.T_1, \cdots$$

Then he defines the derivative of the synthetic function as follows: [9]

$$\frac{d}{dt}[w.T] \equiv \mathcal{D} \quad \Rightarrow \qquad (37)_V \qquad \left[T.\frac{dw}{dt}\right] + \left[w.[w.T]\right] \equiv \mathcal{D} \ ;$$

Here, he defines two vectors as follows:

$$[T] = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} , \quad [w.T] = \begin{bmatrix} w_1T_1 \\ w_2T_2 \\ w_3T_3 \end{bmatrix}$$

then if T_n are independent of time, we can deduce the vectorial form of $(37)_V$:

$$\begin{cases} T_1\frac{dw_1}{dt} + w_2w_3\{T_3 - T_2\} = \mathcal{D}_1, \\ T_2\frac{dw_2}{dt} + w_3w_1\{T_1 - T_3\} = \mathcal{D}_2, \\ T_3\frac{dw_3}{dt} + w_1w_2\{T_2 - T_1\} = \mathcal{D}_3 \end{cases}$$

He states that these are the Euler equations as expressed in tensor form.

4. *Drafts of 'On the dynamical theory of Gases'* by Maxwell

4.1. A progenitor of gas theory after Poisson and Stokes.

(⇓) Even after Poisson, Saint-Venent and Stokes, we can cite the progenitors of microscopically descriptive, hydromechanical equations, which are specializes in gas theories, in which they describe the hydrodynamic equations, and they contribute to fix the tensor and equations of NS, so we have to trace them. cf. Table 2, 3, 4.

Maxwell [29] had presented between late 1865 and early 1866, the original equations calculating his original coefficient, with which his tensor coincides with Poisson and Stokes, and his gas theory prior to Kirchhoff [18] in 1876 and Boltzmann [1] in 1895 as follows: (⇑)

if the motion is not very violent we may also neglect $\frac{\partial}{\partial t}(\rho\xi^2 - p)$ and then we have

$$\xi^2\rho = p - \frac{M}{9k\rho\Theta_2}p\Big(2\frac{du}{dx} - \frac{dv}{dy} - \frac{dw}{dz}\Big) \tag{9}$$

which similar expressions for $\eta^2\rho$ and $\zeta^2\rho$. By transformation of coordinates we can easily obtain the expressions for $\xi\eta\rho$, $\eta\zeta\rho$ and $\zeta\xi\rho$. They are of the form

$$\zeta\xi\rho = -\frac{M}{6k\rho\Theta_2}p\Big(\frac{dv}{dz} + \frac{dw}{dy}\Big) \tag{10}$$

Bei Benutzung dieses Resultates und bei Berücksichtigung der Constanz der Componenten von T nach den mit dem Körper bewegten Axen nimmt die Gleichung $(32)_V$ ($\frac{d}{dt}[w.T] = \mathcal{D}$) die Form an

$$(37)_V \qquad \left[T.\frac{dw}{dt}\right] + \left[w.[w.T]\right] \# \mathcal{D} \ ;$$

es ist dabei zu beachten, daß dieselbe über die Richtungen, nach denen die Componenten der in ihnen auftretenden Vectoren zu nehmen sind, noch weite Freiheit läßt.

Der wichtigste Fall ist der, daß jene Richtungen in die eine Seite der Tensoren T_1, T_2, T_3 - die Hauptträgheitsaxen des Körpers - fallen. Hier reducieren sich nach $(19)_V$ die Componenten von $[w.T]$ auf w_1T_1, w_2T_2, w_3T_3, und es folgt, da die T_h von der Zeit unabhängig sind, aus $(37)_V$,

$$\begin{cases} T_1\frac{dw_1}{dt} + w_2w_3\{T_3 - T_2\} = \mathcal{D}_1, \\ T_2\frac{dw_2}{dt} + w_3w_1\{T_1 - T_3\} = \mathcal{D}_2, \\ T_3\frac{dw_3}{dt} + w_1w_2\{T_2 - T_1\} = \mathcal{D}_3 \end{cases}$$

Das sind die Eulerschen Gleichungen. [45, §11, pp.14-15.]

[9](⇓) By #, Voigt means ≡, i.e. equality by definition.

$$\begin{bmatrix} \rho\overline{\xi_0^2} & \rho\overline{\xi_0\eta_0} & \rho\overline{\xi_0\zeta_0} \\ \rho\overline{\xi_0\eta_0} & \rho\overline{\eta_0^2} & \rho\overline{\eta_0\zeta_0} \\ \rho\overline{\xi_0\zeta_0} & \rho\overline{\zeta_0\eta_0} & \rho\overline{\zeta_0^2} \end{bmatrix} = \begin{bmatrix} X_x & X_y & X_z \\ Y_x & Y_y & Y_z \\ Z_x & Z_y & Z_z \end{bmatrix} = \begin{bmatrix} P_1 & T_3 & T_2 \\ T_3 & P_2 & T_1 \\ T_2 & T_1 & P_3 \end{bmatrix},$$

Having thus obtained the values of the pressures in different directions we may substitute them in the equation of motion

$$\begin{cases} \rho\frac{\partial u}{\partial t} + \frac{d}{dx}(\rho\xi^2) + \frac{d}{dy}(\rho\xi\eta) + \frac{d}{dz}(\rho\xi\zeta) = X\rho, \\ \rho\frac{\partial v}{\partial t} + \frac{d}{dx}(\rho\xi\eta) + \frac{d}{dy}(\rho\eta^2) + \frac{d}{dz}(\rho\eta\zeta) = Y\rho, \\ \rho\frac{\partial w}{\partial t} + \frac{d}{dx}(\rho\xi\zeta) + \frac{d}{dy}(\rho\zeta\eta) + \frac{d}{dz}(\rho\zeta^2) = Z\rho. \end{cases} \tag{11}$$

This becomes as follows :

$$\begin{cases} \rho\frac{\partial u}{\partial t} + \frac{dp}{dx} - \frac{pM}{6k\rho\Theta_2}\left[\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + \frac{1}{3}\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)\right] = \rho X, \\ \rho\frac{\partial v}{\partial t} + \frac{dp}{dy} - \frac{pM}{6k\rho\Theta_2}\left[\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} + \frac{1}{3}\frac{d}{dy}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)\right] = \rho Y, \\ \rho\frac{\partial w}{\partial t} + \frac{dp}{dz} - \frac{pM}{6k\rho\Theta_2}\left[\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2} + \frac{1}{3}\frac{d}{dz}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)\right] = \rho Z. \end{cases} \tag{12}$$

Maxwell states as follows:

> This is the equation of motion in the direction of x. The other equations may be written down by symmetry. The form of the equations is identical
> - with that deduced by Poisson [10] from the theory of elasticity by supposing the strain to be constantly relaxed at the given rate
> - and the ratio of the coefficients of ∇^2 to $\frac{d}{dx}\frac{1}{\rho}\frac{\partial\rho}{\partial t}$ agrees with that given by Professor Stokes, [11] which means (12) equals $(12)_S$.
>
> The quantity $\frac{pM}{6k\rho\Theta_2}$ is the coefficient of viscosity or of internal friction and is denoted by μ in the writings of Professor Stokes and in my paper on the Viscosity of Air and other Gases. [30, pp.261-262].

4.2. **Law of Volumes.**

In late 1865 or early 1866, Maxwell proposed this paper. It was likely that Boltzmann[12] had got his idea from this paper.

> u, v, w are the components of the mean velocity of all the molecules which are at a given instant in a given element of volume, hence there is no motion of translation. ξ, η, ζ are the components of the relative velocity of one of these molecules with respect to the mean velocity, the 'velocity of agitation of molecules'.
>
> In the case of a single gas in motion let Q be the total energy of a single molecule then

$$Q = \frac{1}{2}M\Big\{(u+\xi)^2 + (v+\eta)^2 + (w+\zeta)^2 + \beta(\xi^2+\eta^2+\zeta^2)\Big\}$$

and

$$\frac{\delta Q}{\delta t} = M(uX + vY + wZ).$$

The general equation becomes

$$\begin{aligned} &\frac{1}{2}\rho\frac{\partial}{\partial t}\Big\{u^2+v^2+w^2+(1+\beta)(\xi^2+\eta^2+\zeta^2)\Big\} \\ &+\ \frac{d}{dx}(u\rho\xi^2 + v\rho\xi\eta + w\rho\xi\zeta) + \frac{d}{dy}(u\rho\xi\eta + v\rho\eta^2 + w\rho\eta\zeta) + \frac{d}{dz}(u\rho\xi\zeta + v\rho\eta\zeta + w\rho\zeta^2) \\ &+\ \frac{1}{2}\frac{d}{dx}(1+\beta)\rho\xi(\xi^2+\eta^2+\zeta^2) + \frac{1}{2}\frac{d}{dy}(1+\beta)\rho\eta(\xi^2+\eta^2+\zeta^2) + \frac{1}{2}\frac{d}{dz}(1+\beta)\rho\zeta(\xi^2+\eta^2+\zeta^2) \\ &=\ \rho(uX + vY + wZ). \end{aligned}$$

10(⇓) The Equation(9) in [35, p.139], which we cite as (6) $(7\text{-}9)_{Pf}$ above.

11(⇓) Stokes [42]

12(⇓) 1844-1906.

Substituting the values of $\rho X,\ \rho Y,\ \rho Z$

$$\begin{aligned}&\frac{1}{2}\rho\frac{\partial}{\partial t}(1+\beta)(\xi^2+\eta^2+\zeta^2)\\ +&\ \rho\xi^2\frac{du}{dx}+\rho\eta^2\frac{dv}{dy}+\rho\zeta^2\frac{dw}{dz}+\rho\eta\zeta\Big(\frac{dv}{dz}+\frac{dw}{dy}\Big)+\rho\zeta\xi\Big(\frac{dw}{dx}+\frac{du}{dz}\Big)+\rho\xi\eta\Big(\frac{du}{dy}+\frac{dv}{dx}\Big)\\ +&\ \frac{1}{2}\rho(1+\beta)(\xi^2+\eta^2+\zeta^2)\Big(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\Big)\\ =&\ 0.\end{aligned}$$

Deviding by ρ of both hand-side,

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial t}(1+\beta)(\xi^2+\eta^2+\zeta^2)\\ +&\ \xi^2\frac{du}{dx}+\eta^2\frac{dv}{dy}+\zeta^2\frac{dw}{dz}+\eta\zeta\Big(\frac{dv}{dz}+\frac{dw}{dy}\Big)+\zeta\xi\Big(\frac{dw}{dx}+\frac{du}{dz}\Big)+\xi\eta\Big(\frac{du}{dy}+\frac{dv}{dx}\Big)\\ +&\ \frac{1}{2}(1+\beta)(\xi^2+\eta^2+\zeta^2)\Big(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\Big)\\ =&\ 0.\end{aligned}$$

If we set $\mathcal{R}\equiv\frac{2}{(1+\beta)}$, then we get the second, linear term of the left hand-side by Maxwell is written by tensor

$$\begin{bmatrix}\rho\xi^2 & \rho\xi\eta & \rho\xi\zeta\\ \rho\xi\eta & \rho\eta^2 & \rho\eta\zeta\\ \rho\xi\zeta & \rho\zeta\eta & \rho\zeta^2\end{bmatrix}=-\mathcal{R}\begin{bmatrix}\frac{\partial u}{\partial x} & \Big(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\Big) & \Big(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\Big)\\ \Big(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\Big) & \frac{\partial v}{\partial y} & \Big(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\Big)\\ \Big(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\Big) & \Big(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\Big) & \frac{\partial w}{\partial z}\end{bmatrix}$$

which is 'general tensor'.

4.3. Determination of the inequality of pressure in a medium.

$$\xi^2\rho=p-\frac{M}{9k\rho\Theta_2}p\Big(2\frac{du}{dx}-\frac{dv}{dy}-\frac{dw}{dz}\Big),\ \eta^2\rho=p-\frac{M}{9k\rho\Theta_2}p\Big(\frac{du}{dx}-2\frac{dv}{dy}-\frac{dw}{dz}\Big),$$

$$\zeta^2\rho=p-\frac{M}{9k\rho\Theta_2}p\Big(\frac{du}{dx}-\frac{dv}{dy}-2\frac{dw}{dz}\Big)\eta\zeta\rho=-\frac{M}{6k\rho\Theta_2}p\Big(\frac{dv}{dz}+\frac{dw}{dy}\Big),$$

$$\xi\eta\rho=-\frac{M}{6k\rho\Theta_2}p\Big(\frac{dv}{dz}+\frac{dw}{dy}\Big),\quad \zeta\xi\rho=-\frac{M}{6k\rho\Theta_2}p\Big(\frac{dw}{dx}+\frac{du}{dz}\Big)$$

Here, the relation of the coefficient between (13) and (14) is the relation between $\xi^2\rho\ (=\eta^2\rho=\zeta^2\rho)$ and $\eta\zeta\rho\ (=\xi\eta\rho=\zeta\xi\rho)$ become $\frac{2}{9}\frac{M}{k\rho\Theta_2}=\frac{1}{6}(1+\frac{1}{3})\frac{M}{k\rho\Theta_2}$. The left hand-side corresponds the coefficients of $\frac{du}{dx},\ \frac{du}{dx},\ \frac{du}{dx}$ on the diagonal of the right hand-side in (13). The right hand-side corresponds with the coefficients of $\frac{d^2u}{dx^2},\ \frac{d^2v}{dy^2}$ and $\frac{d^2w}{dz^2}$ in (14).

Then we can construct the tensor which is completely equal to (27) as follows :

$$\begin{bmatrix}\rho\xi^2 & \rho\xi\eta & \rho\xi\zeta\\ \rho\xi\eta & \rho\eta^2 & \rho\eta\zeta\\ \rho\xi\zeta & \rho\zeta\eta & \rho\zeta^2\end{bmatrix}=\begin{bmatrix}p-\frac{M}{9k\rho\Theta_2}p\Big(2\frac{du}{dx}-\frac{dv}{dy}-\frac{dw}{dz}\Big) & -\frac{M}{6k\rho\Theta_2}p\Big(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\Big) & -\frac{M}{6k\rho\Theta_2}p\Big(\frac{dw}{dx}+\frac{du}{dz}\Big)\\ -\frac{M}{6k\rho\Theta_2}p\Big(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\Big) & p-\frac{M}{9k\rho\Theta_2}p\Big(\frac{du}{dx}-2\frac{dv}{dy}-\frac{dw}{dz}\Big) & -\frac{M}{6k\rho\Theta_2}p\Big(\frac{dv}{dz}+\frac{dw}{dy}\Big)\\ -\frac{M}{6k\rho\Theta_2}p\Big(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\Big) & -\frac{M}{6k\rho\Theta_2}p\Big(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\Big) & p-\frac{M}{9k\rho\Theta_2}p\Big(\frac{du}{dx}-\frac{dv}{dy}-2\frac{dw}{dz}\Big)\end{bmatrix}\quad(13)$$

Having thus obtained the values of the pressures in different directions we may substitute them in the equation of motion.

$$\begin{cases}\rho\frac{\partial u}{\partial t}+\frac{d}{dx}(\rho\xi^2)+\frac{d}{dy}(\rho\xi\eta)+\frac{d}{dz}(\rho\xi\zeta)=X\rho,\\ \rho\frac{\partial v}{\partial t}+\frac{d}{dx}(\rho\xi\eta)+\frac{d}{dy}(\rho\eta^2)+\frac{d}{dz}(\rho\eta\zeta)=Y\rho,\\ \rho\frac{\partial w}{\partial t}+\frac{d}{dx}(\rho\xi\zeta)+\frac{d}{dy}(\rho\zeta\eta)+\frac{d}{dz}(\rho\zeta^2)=Z\rho,\end{cases}$$

which become the following equations that are completely equal to $(185)_B$

$$\begin{cases} \rho\frac{\partial u}{\partial t}+\frac{dp}{dx}-\frac{pM}{6k\rho\Theta_2}\left\{\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}+\frac{1}{3}\frac{d}{dx}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)\right\}=X\rho, \\ \rho\frac{\partial v}{\partial t}+\frac{dp}{dy}-\frac{pM}{6k\rho\Theta_2}\left\{\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}+\frac{1}{3}\frac{d}{dy}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)\right\}=Y\rho, \\ \rho\frac{\partial w}{\partial t}+\frac{dp}{dz}-\frac{pM}{6k\rho\Theta_2}\left\{\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+\frac{d^2w}{dz^2}+\frac{1}{3}\frac{d}{dz}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)\right\}=Z\rho \end{cases} \tag{14}$$

Here, it tells of the equivalent in the structure between (13) and (14). If we set $\mathcal{R} \equiv \frac{Mp}{6k\rho\Theta_2}$, then these equations are completely equal to $(221)_B$ by Boltzmann. These facts state that Boltzmann had got his idea of special form of hydromechanics from Maxwell.

4.3.1. *'Lectures on Gas Theory'* **and** *Lectures on Heat Theory* **by Kirchhoff.**

We introduce *'Lectures on Gas Theory'* by Kirchhoff [18, pp.156-172]. He stated his theory citing only Maxwell in 1868 basing on Maxwell's theory as follows :

> Wir wenden uns jetzt zur Betrachtung eines Gases, das nicht in Ruhe ist, und folgen dabei der Maxwell'schen Darstellung.

He says : "We turn here into the investigation of a gas, which is not stable, and follow the description by Maxwell." Afterward, Boltzmann referred many contents of gas theory from both Maxwell and Kirchhoff. For example, Kirchhoff states three assumptions of the number of molecule : we will investigate the change, which these integral operated in a time dt, where the time is infinitesimally small. We show the change by $\frac{\partial(N\overline{Q})}{\partial t}dt$. It consists of three parts :

- the value of Q enlarged by flowing into and flowing out a certain molecule in the parallelepiped in a time dt ;
- The outer force on the molecules, such as gravity operate, make change its velocity ;
- By the collision of each two molecules in the parallelepiped. [19, Lecture 15, p.157]

which Boltzmann cites almost assumptions. In Boltzmann's description about the condition no. 3,

(3) Those of our dn molecules that undergo a *collision* during the time dt will clearly have in general different velocity components after the *collision*.
  - (Decrease :) Their velocity points will therefore be expected, as it were, from the parallelepiped by the *collision*, and thrown into a completely different parallelepiped. The number dn will thereby be *decreased*.
  - (Increase :) On the other hand, the velocity points of m-molecules in other parallelepipeds will be throne into $d\omega$ by *collisions*, and dn will thereby *increase*.
  - (Total increase by *collision* between m-molecules and m_1-molecules :) It is now a question of finding this total increase V_3 experienced by dn during time dt as a result of the *collisions* taking place between any m-molecules and any m_1-molecules.

In 1894, Kirchhoff, in *Lectures on Heat Theory* [19, p.194], stated hydrodynamic equations in incompressible fluid.

$$\begin{cases} \mu\frac{du}{dt}+\frac{\partial}{\partial x}-\frac{1}{3\kappa}\frac{p}{\mu}\left[\Delta u+\frac{1}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right]=\mu X, \\ \mu\frac{dv}{dt}+\frac{\partial p}{\partial y}-\frac{1}{3\kappa}\frac{p}{\mu}\left[\Delta v+\frac{1}{3}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right]=\mu Y, \\ \mu\frac{dw}{dt}+\frac{\partial p}{\partial z}-\frac{1}{3\kappa}\frac{p}{\mu}\left[\Delta z+\frac{1}{3}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right]=\mu Z, \end{cases}$$

$$\frac{1}{\mu}\frac{d\mu}{dt}+\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0.$$

Kirchhoff explains his viscosity term as follows :

> Als solche werden wir annehmen, daß u, v, w in dem Gas dieselben Werthe haben, wie in dem festen Körper, also verschwinden, wenn dieser ruht; und daß die absolute Temperatur im Gas, die $\frac{p}{\mu}$ mal einer Constanten ist, gleich ist der Temperatur des fasten Körpers. ···
>
> Die mit $\frac{1}{\kappa}$ proportionalen Glieder, durch welch unsere Gleichungen sich unterscheiden von den in erster Annäherung geltenden, bedinden die Erscheinungen der *Reibung* und der *Wärmeleitung*. ···
>
> Die Grösse $\frac{1}{3\kappa}\frac{p}{\mu}$ heißt der *Reibungscoefficient*. [19, §3, pp.194-5]

[(transl.) We assume it as such that u, v, w in the gas have each value in the solid, when these move, and that the absolute temperature in gas which is equal to the multiplied by $\frac{p}{\mu}$ of an constant, is equal to the temperature of solid. $\cdots$ The proportional terms with $\frac{1}{\kappa}$, by which our equations are distinguished with one in the first adaption, bring up as the phenomena of viscosity and the heat conduction. $\cdots$ The term $\frac{1}{3\kappa}\frac{p}{\mu}$ is called by viscosity coefficient. $\cdots$]

He introduces the real value of $\frac{1}{3\kappa}\frac{p}{\mu}$ in his following context, which we omit it for lack of space.

5. *'Lectures on Gas theory'* by Boltzmann

In general, according to Ukai [43], we can state the Boltzmann equations as follows: [13]

$$\partial_t f + \mathbf{v}\cdot\nabla_{\mathbf{x}} f = Q(f,g), \quad t>0, \quad \mathbf{x},\mathbf{v}\in\mathbb{R}^n (n\geq 3), \quad \mathbf{x}=(x,y,z), \quad \mathbf{v}=(\xi,\eta,\zeta), \tag{15}$$

$$Q(f,g)(t,x,v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,\sigma)\{g(v_*')f(v') - g(v_*)f(v)\}d\sigma dv_*, \quad g(v_*') = g(t,x,v_*'), \tag{16}$$

$$v' = \frac{v+v_*}{2} + \frac{|v+v_*|}{2}\sigma, \quad v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \quad \sigma\in\mathbb{S}^{n-1} \tag{17}$$

where,

- $f = f(t,\ x,\ v)$ is interpretable as many meanings such as
  - density distribution of a molecule
  - number density of a molecule
  - probability density of a molecule

  at time : t, place : $\mathbf{x}$ and velocity : $\mathbf{v}$.
- $f(v)$ means $f(t,x,v)$ as abbreviating t and x in the same time and place with $f(v')$
- $Q(f,g)$ of the right-hand-side of (15) is the Boltzmann bilinear *collision operator*.
- $\mathbf{v}\cdot\nabla_{\mathbf{x}} f$ is the *transport operator*,
- $B(z,\sigma)$ of the right-hand-side in (16) is the non-negative function of *collision cross-section.*
- $Q(f;g)(t,x,v)$ is expressed in brief as $Q(f)$.
- $(v,\ v_*)$ and $(v',\ v_*')$ are the velocities of a molecule before and after collision.
- According to Ukai [44], the *transport operators* are expressed with two sort of terms like Boltzmann's descriptions : $(114)_B$ and $(115)_B$ including the collision term $\nabla_{\mathbf{v}}\cdot(\mathbf{F}f)$ by exterior force $\mathbf{F}$ as follow : [14]

$$\partial_t f + \mathbf{v}\cdot\nabla_{\mathbf{x}} f + \nabla_{\mathbf{v}}\cdot(\mathbf{F}f) = Q(f) \tag{18}$$

$$Q(f) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,\sigma)\{f(v_*')f(v') - f(v_*)f(v)\}d\sigma dv_* \tag{19}$$

where, $\mathbf{v}\cdot\nabla_{\mathbf{x}} f + \nabla_v\cdot(\mathbf{F}f)$ are *transport operators* operating under the exterior force : $\mathbf{F}(t,x,v) = (F_1,F_2,F_3)$. The right-hand side of (18) is expressed in brief as $Q(f)$ meaning $Q(f)(t,x,v)$.

5.1. Development of partial differential equations for f and F.

We show the Figure 6 in the last page of our paper, which defines the model of the *collision* between the molecule m_1 calling the point of it and the molecule m wich we call the point m. The instant when the molecule m passes vertically throught the disc of m_1 molecule, is defined as *collision.* We show Boltzmann's definitions as follow :

> We fix our attention on the parallelepiped representing all space points whose coordinates lie between the limits [15]
>
> $$(97)_B \quad [x,\ x+dx], \quad [y,\ y+dy], \quad [z,\ z+dz], \quad do = dxdydz$$
>
> We now construct a second rectangular parallelepiped, which include all points whose coordinates lie between the limits
>
> $$(98)_B \quad [\xi,\ \xi+d\xi], \quad [\eta,\ \eta+d\eta], \quad [\zeta,\ \zeta+d\zeta]$$
>
> We set its volume equal to

$$d\xi d\eta d\zeta = d\omega \tag{20}$$

[13](⇓) We refer the *Lecture Note* by S.Ukai: *Boltzmann equations: New evolution of theory, Lecture Note of the Winter School in Kyushu of Non-linear Partial Differential Equations*, Kyushu University, 6-7, November, 2009.

[14](⇓) In the Boltzmann' original equations, they are used with two terms like $(114)_B$, $(115)_B$. We can refer the *General lecture in the autumn meeting of MSJ* by S.Ukai [44] : *The study of Boltzmann equations: past and future*, MSJ, 23, September, 2010.

[15](⇓) $(\cdot)_B$ in the top of the equation or expression means the number cited in Boltzmann[2] in below of our paper.

TABLE 5. The symbols and definitions

no	symbol	defined	content of conformation in modeling of collision. cf. The Fig. 6 in the last page.	cf.	m	m_1
1	$X,\ Y,\ Z$	(21)	The component of accelarating force of a molecule in a coödinate direction.			
2	mX, mY, mZ		The component of the external force acting on any m-molecule.		m	
3	$\xi,\ \eta,\ \zeta$	$(98)_B$	The component of velocity of any m-molecule in a coödinate direction.		m	
4	f	$(99)_B$	$f = f(x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t)$		m	
5	f_1	$(99)_B$	$f_1 = f(x,\ y,\ z,\ \xi_1,\ \eta_1,\ \zeta_1,\ t)$, different only with velocity of f.		m	
6	F	$(100)_B$	$F = F(x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t)$			m_1
7	F_1	$(103)_B$	$F_1 = F(x,\ y,\ z,\ \xi_1,\ \eta_1,\ \zeta_1,\ t)$, different only with velocity of F.			m_1
8	$\xi_1,\ \eta_1,\ \zeta_1$	$(102)_B$	The component of velocity of any m_1-molecule in a coödinate direction.			m_1
9	g	p.116	The moving direction (or velocity) of an m-molecule to an m_1-molecule.	Fig. 6	m	
10	gdt	p.116	The moving distance of an m-molecule to an m_1-molecule during dt.	Fig. 6	m	
11	b	$(104)_B$	The length of a line originated from m_1-molecule, where, b is the smallest possible distance of the two colliding molecules that could be attained if they moved without interaction in straight lines with the velocities they had before the collision. In other words, b is the line P_1P, where P_1 and P are the two points at which m_1 and m would be found at the moment of their closest approach if there were no interraction.	Fig. 6		m_1
12	σ		The limit of the length of a line. $[0,\ \sigma]$.	Fig. 6		m_1
13	ϵ	$(104)_B$	An angle formed between a line b and a line m_1H, where, ϵ is the angle between the two planes through the direction of relative motion, one parallel to P_1P along b, and the other to the abscissa axis.	Fig. 6		m_1
14	$\xi',\ \eta',\ \zeta'$	$(108)_B$	The component of velocity of a molecule after the *collision.*		m	
15	b'	$(109)_B$	The length of a line after the *collision.*	Fig. 6		m_1
16	ϵ'	$(109)_B$	An angle formed between a line b and a line m_1H after the *collision* .	Fig. 6		m_1
17	do : parallelepiped	$(97)_B$	We set $do = dxdydz$ in which the m-molecules lie, and we always call this parallelepiped the parallelepiped do.		m	
18	$d\omega$: parallelepiped of velocity point	(98_B) (20)	We set $d\omega = d\xi d\eta d\zeta$ in which velocity point of the m-molecules lie, and we always call this parallelepiped the parallelepiped $d\omega$.		m	
19	$d\omega_1$	(102_B) (24)	We set $d\omega_1 = d\xi_1 d\eta_1 d\zeta_1$ as well as $d\omega$, in which velocity point of the m_1-molecules lie, and we always call this parallelepiped the parallelepiped $d\omega_1$.			m_1
20	dn	$(99)_B$	The m-molecules that are in do at time t and whose velocity points lie in $d\omega$ at the same time will again be called the specified molecules, or the "dn molecules." $dn = f(x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t)dod\omega = fdod\omega$		m	
21	dn'	$(99)'_B$	The number of m-molecules that satisfy the conditions $(97)_B$ and $(98)_B$ at time $t+dt$. $dn' = f(x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t+dt)dod\omega$		m	
22	dN	$(100)_B$	The number of m_1-molecules that satisfy the conditions $(97)_B$ and $(98)_B$ at time t. $dN = F(x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t)dod\omega = Fdod\omega$			m_1
23	dN_1	$(103)_B$	$dN_1 = F(x,\ y,\ z,\ \xi_1,\ \eta_1,\ \zeta_1,\ t)dod\omega = F_1dod\omega_1$			m_1
24	ν_1	$(107)_B$	The number of all *collisions* of our dn molecules during dt with m_1-molecules.		m	m_1
25	ν_2	$(106)_B$	The number of m-points that pass an m_1-point at any distance less than σ during dt.		m	m_1
26	ν_3	$(105)_B$	The number of *collisions* between m-molecules and m_1-molecules.		m	m_1
27	V_1	(22)	The increase which dn experiences as a result of motion of the molecules during time dt, where all m-molecules whose velocity points lie in $d\omega$ move in the x-direction with velocity ξ, in the y-direction with velocity η, and in the z-direction with velocity ζ.	$A_2(\varphi)$	m	
28	V_2	(23)	As a result of the action of *external forces*, the velocity components of all the molecules change with time, and hence the velocity points of the molecules in do will move.	$A_3(\varphi)$	m	
28	i_1	$(111)_B$	The *total increase* experienced by dn as a result of *collisions* of m-molecules with m_1-molecules.		m	m_1
30	V_3	$(112)_B$	The *net increase* experienced by dn as a result of *collisions* of m-molecules with m_1-molecules. $V_3 = i_1 - \nu_1$.	$A_4(\varphi)$	m	m_1
31	V_4	$(113)_B$	The increment experienced by dn as a result of *collisions* of m or m_1-molecules with *each other.*	$A_5(\varphi)$	m	m_1
32	$\varphi,\ \sum_{d\omega,do}\varphi$	$(116)_B$	$\varphi = \varphi(x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t)$, $\sum_{d\omega,do}\varphi \equiv \varphi fdod\omega$, multipling the number $fdod\omega$ by φ		m	
33	$\Phi,\ \sum_{d\omega,do}\Phi$	$(117)_B$	$\Phi = \Phi(x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t)$, $\sum_{d\omega,do}\Phi \equiv \Phi Fdod\omega$, multipling the number $Fdod\omega$ by Φ		m	
34	$\Phi_1,\ \sum_{d\omega,do}\Phi_1$	$(117)_B$	$\Phi_1 = \Phi(x,\ y,\ z,\ \xi_1,\ \eta_1,\ \zeta_1,\ t)$, $\sum_{d\omega,do}\Phi_1 \equiv \Phi_1 F_1 dod\omega_1$, multipling the number $F_1dod\omega$ by Φ_1			m_1
35	$A_1(\varphi)$	$(121)_B$	The effect of explicit dependance of φ on t.			
36	$A_2(\varphi)$	$(122)_B$	The effect of *the motion* of the molecules.	V_1	m	
37	$A_3(\varphi)$	$(123)_B$	The effect of *external forces.*	V_2	m	
38	$A_4(\varphi)$	$(124)_B$	The effect of collisions of m-molecules with m_1-molecules.	V_3	m	m_1
39	$A_5(\varphi)$	$(125)_B$	The effect of collisions of m-molecules with *each other.*	V_4	m	
40	$B_1(\varphi)$	$(127)_B$	The total effect in ω of explicit dependance of φ on t.			
41	$B_2(\varphi)$	$(128)_B$	The effect in ω of *the motion* of the molecules.	V_1	m	
42	$B_3(\varphi)$	$(129)_B$	The effect in ω of *external forces.*	V_2	m	
43	$B_4(\varphi)$	$(134)_B$	The effect in ω of collisions of m-molecules with m_1-molecules.	V_3	m	m_1
44	$B_5(\varphi)$	$(139)_B$	The effect in ω of collisions of m-molecules with *each other.*	V_4	m	
44	$\{C_n(\varphi)\}_1^5$	$(125)_B$	The effect in ω and o as the same as $\{A_n(\varphi)\}_1^5$ or $\{B_n(\varphi)\}_1^5$			

and we call it the parallelepiped $d\omega$. The molecules that are in do at the time t and whose velocity points lie in $d\omega$ at the same time will again be called the specified molecules, or the "dn molecules." Their number is clearly proportional to the product $do \cdot d\omega$. Then all volume elements immediately adjacent to do find themselves subject to similar conditions, so that in a parallelepiped twice as large there will be twice as many molecules. We can therefore set this number equal to

$$(99)_B \quad dn = f(x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t)dod\omega = fdod\omega$$

Similarly the number of m_1-molecules that satisfy the conditions $(97)_B$ and $(98)_B$ at time t will be :

$$(100)_B \quad dN = F(x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t)dod\omega = Fdod\omega$$

The two functions f and F completely characterize the state of motion, the mixing ratio, and the velocity distribution at all places in the gas mixture. We shall allow a very short time dt to elapse, and during this time we keep the size and position of do and $d\omega$ completely unchanged. The number of m-molecules that satisfy the conditions $(97)_B$ and $(98)_B$ at time $t + dt$ is, according to Equation $(99)_B$,

$$dn' = f(x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t+dt)dod\omega = fdod\omega$$

and the total increase experienced by dn during time dt is

$$(101)_B \quad dn' - dn = \frac{\partial f}{\partial t} do\ d\omega\ dt.$$

$\xi,\ \eta,\ \zeta$ are the rectangular coödinates of the velocity point. Although this is only an imaginary point, still it moves like the molecule itself in space. Since $X,\ Y,\ Z$ are the components of the accelerating force,[16] we have:

$$\frac{d\xi}{dt} = X, \quad \frac{d\eta}{dt} = Y, \quad \frac{d\zeta}{dt} = Z \tag{21}$$

5.2. **Four different causes bringing up increase of dn.**

Boltzmann explains an increase of dn as a result of the following *four different causes* of $V_1,\ V_2,\ V_3$ and V_4 :

- V_1 : increment by *transport* through do
- V_2 : increment by *transport* of external force
- V_3 : increment as a result of *collisions* of m-molecules with m_1-molecules
- V_4 : increment by *collision* of molecules with each other

We extract an outline by the Boltzmann [1] as follows :

The number dn experiences an increase as a result of *four different causes.*

(1) (V_1 : increase going out through do ;) All m-molecules whose velocity points lie in $d\omega$ move in the x-direction with velocity ξ, in the y-direction with velocity η, and in the z-direction with velocity ζ.

Hence through the left of the side of the parallelepiped do facing the negative abscissa direction there will enter during time dt as many molecules satisfying the condition (98_B) as may be found, at the beginning of dt, in a parallelepiped of base $dydz$ and height ξdt,[17] viz.

$$\xi \cdot f(x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t)dydzd\omega dt$$

molecules. Likewise, for the number of $m-$molecules that satisfying (98_B) and go out through the opposite face of do during time dt, the value:

$$\xi \cdot f(x + dx,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t)dydzd\omega dt$$

16(⇓) Da $X,\ Y,\ Z$ die Componenten der beschleunigenden Kraft sind, so ist: $\cdots$ Boltzmann [2, p.103].

17(⇓) ξ : the x-direction with velocity multiplied by dt becomes the length of a edge of which consists a parallelepiped with a base $dydz$.

By similar arguments for the four other sides of the parallelepiped, one finds that during time dt,

$$-\Big(\xi\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y}+\zeta\frac{\partial f}{\partial z}\Big)do\cdot d\omega\ dt$$

more molecules satisfying (98_B) enter do than leave it. This is therefore the increase V_1 which dn experiences as a result of motion of the molecules during time dt.

$$V_1=-\Big(\xi\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y}+\zeta\frac{\partial f}{\partial z}\Big)do\ d\omega\ dt \tag{22}$$

(2) (V_2 : increase by external force ;) As a result of the action of external forces, the velocity components of all the molecules change with time, and hence the velocity points of the molecules in do will move. Some velocity points will leave $d\omega$, others will come in, and since we always include in the number dn only those molecules whose velocity points lie in $d\omega$, dn likewise be changed for this reason.

$$V_2=-\Big(X\frac{\partial f}{d\xi}+Y\frac{\partial f}{\partial y}+Z\frac{\partial f}{\partial z}\Big)do\ d\omega\ dt \tag{23}$$

Boltzmann defines the effects of *collisions* as follows :

(3) (V_3 : increase as a result of *collisions* of m-molecules with m_1-molecules ;) Those of our dn molecules that undergo a *collision* during the time dt will clearly have in general different velocity components after the *collision*.

- (Decrease :) Their velocity points will therefore be expected, as it were, from the parallelepiped by the *collision*, and thrown into a completely different parallelepiped. The number dn will thereby be *decreased*.
- (Increase :) On the other hand, the velocity points of m-molecules in other parallelepipeds will be throne into $d\omega$ by *collisions*, and dn will thereby *increase*.
- (Total increase by *collision* between m-molecules and m_1-molecules :) It is now a question of finding this total increase V_3 experienced by dn during time dt as a result of the *collisions* taking place between any m-molecules and any m_1-molecules.

For this purpose we shall fix our attention on a very small fraction of the total number ν_1 of *collisions* undergone by our dn molecules during time dt with m_1-molecules. We construct a third parallelepiped which includes all points whose coordinates lie between the limits

$$(102)_B\quad [\xi_1,\ \xi_1+d\xi_1],\quad [\eta_1,\ \eta_1+d\eta_1],\quad [\zeta_1,\ \zeta_1+d\zeta_1]$$

Its volume is

$$d\omega_1=d\xi_1 d\eta_1 d\zeta_1 \tag{24}$$

It constitutes the parallelepiped $d\omega_1$. By analogy with Equation $(100)_B$, the number of m_1-molecules in do whose velocity points lie in $d\omega_1$ at time t is :

$$(103)_B\quad dN_1=F_1 dod\omega_1,$$

where F_1 is an abbreviation for $F(x,\ y,\ z,\ \xi_1,\ \eta_1,\ \zeta_1)$.

Boltzmann difines a passage of an m-point by an m_1-point as follows :

(a) (How to pass :) We define a passage of an m-point by an m_1-point as that instant of time when distance between the points has its smallest value ; thus m would pass through the plane through m_1 perpendicular to the direction g, if no interaction took place between the two molecules.

(b) (ν_2 : the number of passages of an m-point by an m_1-point :) Hence, ν_2 is equal to the number of passages of an m-point by an m_1-point that occurs during time dt, such that the smallest distance between the two molecules is less than σ.
(c) (A plane E :) In order to find this number, we draw through each m_1-point a plane E moving with m_1, perpendicular to the direction of g, and a line G, which parallel to this direction.
(d) (When a passage ends :) As soon as an m-point crosses E, a passage take place between it and the m_1-point.
(e) (A line m_1X :) We draw through each m_1-point a line m_1X parallel to the positive abscissa direction and similarly directed.
(f) (Half-plane :) The half-plane bounded by G, which contains the latter line, cuts E in the line m_1H, which of course again contains each m_1-point.
(g) (b and ϵ :) Furthermore, we draw from each m_1-point in each of the plane E a line of length b, which forms an angle ϵ with the line m_1H.
(h) (Rectangles of surface area R formed by b and ϵ :) All points of the plane E for which b and ϵ lie between the limits

$$(104)_B \quad [b,\ b+db], \quad [\epsilon,\ \epsilon+d\epsilon]$$

form a rectangle of surface area $R = bdbd\epsilon$.

In Figure 6 [18] the intersections of all these lines with a sphere circumscribed about m_1 are shown. The large circle (shown as an ellipse) lies in the plane E ; the circular arc GXH lies in the half-plane defined above. In each of planes E, an equal and identically situated rectangle will be found. We consider for the moment only those passages of an m-point by an m_1-point in which the first point penetrates one of the rectangles R.

$$\Pi = Rgdt = \underbrace{bdbd\epsilon}_{R}\, gdt, \quad \sum \Pi = dN_1\Pi = \underbrace{F_1 dod\omega_1}_{dN_1\ (103)_B}\ \underbrace{gbdbd\epsilon dt}_{\Pi}$$

Since these volumes are infinitesimal, and lie infinitely close to the point with coordinates $x,\ y,\ x$, then by analogy with Equation $(99)_B$ the number of m-points (i.e., m-molecules whose velocity points lie in $d\omega$) that are initially in the volumes $\sum \Pi$ is equal to :

$$(105)_B \quad \nu_3 = fd\omega \sum \Pi = fF_1 dod\omega d\omega_1 gbdbd\epsilon dt$$

This is at the same time the number of m-points that pass an m_1-point during time dt at a distance between b and $b+db$, in such a way that the angle ϵ lie between ϵ and $\epsilon + d\epsilon$.

By ν_2 we mean the number of m-points that pass an m_1-point at any distance less than σ during dt. We find ν_2 by integrating the differential expression ν_3 over ϵ from 0 to 2π, and over b from 0 to σ.

$$(106)_B \quad \nu_2 = \int_0^\sigma db \int_0^{2\pi} \nu_3 d\epsilon = dod\omega d\omega_1 dt \int_0^\sigma db \int_0^{2\pi} d\epsilon\ g \cdot b \cdot f \cdot F_1 \cdot$$

The number denoted by ν_1 of all *collisions* of our dn molecules during dt with m_1-molecules is therefore found by integrating over the three variable $\xi_1,\ \eta_1,\ \zeta_1$ whose differentials occur in $d\omega_1$, from $-\infty$ to $+\infty$; we indicate this a single integral sign :

$$(107)_B \quad \nu_1 = \int_{-\infty}^{\infty} \nu_2 d\omega_1 = do \cdot d\omega \cdot dt \int_{-\infty}^{\infty} d\omega_1 \int_0^\sigma db \int_0^{2\pi} fF_1 gbd\epsilon$$

We shall consider again those *collisions* between m-molecules and m_1-molecules, whose number was denoted by ν_3 and is given by Equation $(105)_B$.

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[18] (⇓) We show this Figure 6 in the last page of our paper citing [2, p.107], which is equal to [1, p.117], however, we must correct the symbol R by H of [1, p.117].

These are the *collisions* that occur in unit time in the volume element *do* in such a way the following conditions are satisfied :

- The velocity components of the m-molecules and the m_1-molecules lie between the limits $(98)_B$ and $(102)_B$, respectively, before the interaction begins.
- We denote by b the closest distance of approach that would be attained if the molecules did not interact but retained the velocities they had before the *collision.*

The *total increment* i_1 experienced by dn as a result of *collisions* of m-molecules with m_1-molecules is founded by integrating over ϵ from 0 to 2π, over b from 0 to σ, and over ξ_1, η_1, ζ_1 from $-\infty$ to $+\infty$. We shall write the result of this integration in the form :

$$(111)_B \quad i_1 = dod\omega dt \iint_0^{\sigma} \int_0^{2\pi} f' F_1' g b d\omega_1 db d\epsilon$$

Of course we cannot perform explicitly the integration with respect to b and ϵ since the variable ξ', η', ζ' and ξ_1', η_1', ζ_1' occurring in f' and F_1' are functions of (ξ, η, ζ, ξ_1', η_1', ζ_1', b and ϵ), which cannot be computed until the force law is given.[19]

The difference $i_1 - \nu_1$ expresses the *net increase* of dn during time dt as a result of *collisions* of m-molecules with m_1-molecules. It is therefore the *total increase* V_3 experienced by dn as a result of these *collisions*, and one has

$$(112)_B \quad V_3 = i_1 - \nu_1 = dod\omega dt \iint_0^{\sigma} \int_0^{2\pi} (f' F_1' - f F_1) d\omega_1 db d\epsilon$$

(4) (V_4 : increment by collision of molecules with each other ;) The increment V_4 experienced by dn as a result of *collisions* of m-molecules with each other is found from Equation $(112)_B$ by a simple permutation. One now uses ξ_1, η_1, ζ_1 and ξ_1', η_1', ζ_1' for the velocity components of the other m-molecule *before and after the collision,* respectively, and one writes f_1 and f_1' for

$$f_1 = f(x,\ y,\ z,\ \xi_1,\ \eta_1,\ \zeta_1,\ t) \quad \text{and} \quad f_1' = f(x,\ y,\ z,\ \xi_1',\ \eta_1',\ \zeta_1',\ t)$$

Then :

$$(113)_B \quad V_4 = dod\omega dt \iint_0^{\infty} \int_0^{2\pi} (f' f_1' - f f_1) g b d\omega_1 db d\epsilon$$

5.3. Formulation of Boltzmann's transport equations.

According to Boltzmann[2, pp.110-115], [20] his equations (so-called *transport equations*) are the following : [21]

Since now $V_1 + V_2 + V_3 + V_4$ is equal to the increment $dn' - dn$ of dn during time dt, and this according to Equation $(101)_B$ must be equal to $\frac{\partial f}{\partial t} dod\omega dt$, one obtains on substituting all the appropriate value and deviding by $dod\omega dt$ the following partial differential equation for the function f :

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[19](⇓) Hier kann die Integration nach b und ϵ natürlich nicht mehr sofort aus geführt werden, da die in f' and F_1' vorkommen den Variabeln ξ', η', ζ' und ξ_1', η_1', ζ_1' Function von ξ, η, ζ, ξ_1', η_1', ζ_1', b und ϵsind, welche nur berechnet werden können, wenn Virkungsgesetz der während eines Zusammenstosses wirksamen Kräfte gegeben ist. [2, p.112].

[20](⇓) Boltzmann(1844-1906) had put the date in the foreword to part I as September in 1895, part II as August in 1898.

[21](⇓) We mean the equation number in the left-hand side with $(\cdot)_B$ the citations from the Boltzmann[2] or [1]. We state only the symbol $\int$ instead of $\int_{-\infty}^{\infty}$. cf. $(107)_B$.

$$(114)_B \quad \frac{\partial f}{\partial t} + \underbrace{\xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \zeta\frac{\partial f}{\partial z}}_{V_1} + \underbrace{X\frac{\partial f}{\partial x} + Y\frac{\partial f}{\partial y} + Z\frac{\partial f}{\partial z}}_{V_2}$$

$$= \underbrace{\iint_0^\infty \int_0^{2\pi} (f'F_1' - fF_1)gb\, d\omega_1\, db\, d\epsilon}_{V_3} + \underbrace{\iint_0^\infty \int_0^{2\pi} (f'f_1' - ff_1)gb\, d\omega_1\, db\, d\epsilon}_{V_4}$$

$$= \underbrace{\iint_0^\infty \int_0^{2\pi} \Big[(f'F_1' - fF_1) + (f'f_1' - ff_1)\Big]gb\, d\omega_1\, db\, d\epsilon}_{V_3+V_4}$$

Similarly we obtain the equation of F :

$$(115)_B \quad \frac{\partial F_1}{\partial t} + \underbrace{\xi_1\frac{\partial F_1}{\partial x} + \eta_1\frac{\partial F_1}{\partial y} + \zeta_1\frac{\partial F_1}{\partial z}}_{V_1} + \underbrace{X_1\frac{\partial F_1}{\partial x} + Y_1\frac{\partial F_1}{\partial y} + Z_1\frac{\partial F_1}{\partial z}}_{V_2}$$

$$= \underbrace{\iint_0^\infty \int_0^{2\pi} (f'F_1' - fF_1)gb\, d\omega_1\, db\, d\epsilon}_{V_3} + \underbrace{\iint_0^\infty \int_0^{2\pi} (F'F_1' - FF_1)gb\, d\omega_1\, db\, d\epsilon}_{V_4}$$

$$= \underbrace{\iint_0^\infty \int_0^{2\pi} \Big[(f'F_1' - fF_1) + (F'F_1' - FF_1)\Big]gb\, d\omega_1\, db\, d\epsilon}_{V_3+V_4}$$

where,

$$\begin{cases} f = f(x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t), \quad f_1 = f(x,\ y,\ z,\ \xi_1,\ \eta_1,\ \zeta_1,\ t), \quad f_1' = f(x,\ y,\ z,\ \xi_1',\ \eta_1',\ \zeta_1',\ t), \\ F = F(x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t), \quad F_1 = F(x,\ y,\ z,\ \xi_1,\ \eta_1,\ \zeta_1,\ t), \quad F_1' = F(x,\ y,\ z,\ \xi_1',\ \eta_1',\ \zeta_1',\ t) \end{cases} \tag{25}$$

Namely, we can verify $(114)_B$ for f :

TABLE 6. Combination of function before and after collision

no	item	V_3 before	V_3 after	f of V_4 before	f of V_4 after	F of V_4 before	F of V_4 after
1	function of m_1	f	f'	f	f'	F	F'
2	function of m	F_1	F_1'	f_1	f_1'	F_1	F_1'
3	increment		$f'F_1' - fF_1$		$f'f_1' - ff_1$		$F'F_1' - FF_1$

$$\frac{V_1+V_2+V_3+V_4}{dod\omega dt} = \frac{\partial f}{\partial t} = -\underbrace{\Big(\xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \zeta\frac{\partial f}{\partial z}\Big)}_{V_1} - \underbrace{\Big(X\frac{\partial f}{d\xi} + Y\frac{\partial f}{\partial y} + Z\frac{\partial f}{\partial z}\Big)}_{V_2}$$

$$+ \underbrace{\iint_0^\infty \int_0^{2\pi} (f'F_1' - fF_1)gb \cdot d\omega_1 dbd\epsilon}_{V_3} + \underbrace{\iint_0^\infty \int_0^{2\pi} (f'f_1' - ff_1)gb \cdot d\omega_1 dbd\xi.}_{V_4}$$

Similarly we obtain $(115)_B$ for F.

$$\frac{V_1+V_2+V_3+V_4}{dod\omega dt} = \frac{\partial F_1}{\partial t} = -\Big(\xi\frac{\partial F_1}{\partial x} + \eta\frac{\partial F_1}{\partial y} + \zeta\frac{\partial F_1}{\partial z}\Big) - \Big(X\frac{\partial F_1}{d\xi} + Y\frac{\partial F_1}{\partial y} + Z\frac{\partial F_1}{\partial z}\Big)$$

$$+ \iint_0^\infty \int_0^{2\pi} (f'F_1' - fF_1)gb \cdot d\omega_1 dbd\epsilon + \iint_0^\infty \int_0^{2\pi} (F'F_1' - FF_1)gb \cdot d\omega_1 dbd\epsilon.$$

(⇓) Here, we can confirm the identity with the today's description of the Boltzmann equations (15) and (16) :

$$\partial_t f + \underbrace{\mathbf{v}\cdot\nabla_{\mathbf{x}} f}_{V_1} + \underbrace{\mathbf{w}\cdot\nabla_{\mathbf{v}} f}_{V_2} = \underbrace{Q(f,g)}_{V3,\ V4}, \quad \partial_t F + \underbrace{\mathbf{v}\cdot\nabla_{\mathbf{x}} F}_{V_1} + \underbrace{\mathbf{w}\cdot\nabla_{\mathbf{v}} F}_{V_2} = \underbrace{Q(F,G)}_{V3,\ V4},$$

$$Q(f,g)(t,x,v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,\sigma)\{g(v_*')f(v') - g(v_*)f(v)\}d\sigma dv_*, \quad g(v_*') = g(t,x,v_*'), \text{ etc.}$$

$$t>0, \quad \mathbf{x},\mathbf{v},\mathbf{w} \in \mathbb{R}^n (n \geq 3), \quad \mathbf{x} = (x,y,z), \quad \mathbf{v} = (\xi,\eta,\zeta), \quad \mathbf{w} = (X,\ Y,\ Z).$$

In the case of (18) and (19)

$$\partial_t f + \underbrace{\mathbf{v}\cdot\nabla_{\mathbf{x}} f}_{V_1} + \underbrace{\nabla_{\mathbf{v}}\cdot(\vec{F} f)}_{V_2} = \underbrace{Q(f)}_{V_3, V_4}$$

$$Q(f) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v - v_*, \sigma)\{f(v_*')f(v') - f(v_*)f(v)\}d\sigma dv_*$$

5.4. **Time-derivatives of sums over all molecules in a region.**

Let φ be an arbitrary function of $x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t$. The value obtained by substituting therein the actual coödinates and velocity components of a particular molecule at time t will be called the value of φ corresponding to that molecule at time t. The sum of all values of φ corresponding to all the m-molecules that lie in the parallelepiped do and whose velocity points lie in the parallelepiped $d\omega$ at time t is obtained by multiplying φ by the number $f do d\omega$ of those molecules. We denote it by $(116)_B$.

Similarly we choose for the second kind of gas any other arbitrary function Φ of $x,\ y,\ z,\ \xi,\ \eta,\ \zeta,\ t$ and denote by $(117)_B$. The sum of the values of Φ corresponding to all the m_1-molecules lying in do whose velocity points lie in $d\omega_1$. Φ_1 is the abbreviation for $\Phi(x,\ y,\ z,\ \xi_1,\ \eta_1,\ \zeta_1,\ t)$. [1, §.17, pp.123-124].

5.5. **General form of the hydrodynamic equations.**

As the general expressions for fluid mechanics, he states that when we substitute for $\frac{\partial f}{\partial t}$ its value from Equation $(114)_B$, it turns into $(120)_B$, $(126)_B$, $(140)_B$, a sum of five terms, each of which has its own physical meaning, as follows:

$$\begin{cases} (116)_B\ \sum_{d\omega,do}\varphi \equiv \varphi f do d\omega, & (120)_B\ \frac{\partial}{\partial t}\sum_{d\omega,do}\varphi = \left(f\frac{\partial\varphi}{\partial t} + \varphi\frac{\partial f}{\partial t}\right)do d\omega = \left[\sum_{n=1}^{5} A_n(\varphi)\right]do d\omega, \\ (117)_B\ \sum_{d\omega,do}\Phi \equiv \Phi F do d\omega_1, & \sum_{d\omega,do}\Phi_1 = \Phi_1 F_1 do d\omega_1, \\ (118)_B\ \sum_{\omega,do}\varphi \equiv do\int\varphi f d\omega, & (126)_B\ \frac{\partial}{\partial t}\sum_{\omega,do}\varphi = do\int\left(f\frac{\partial\varphi}{\partial t} + \varphi\frac{\partial f}{\partial t}\right)d\omega = \left[\sum_{n=1}^{5} B_n(\varphi)\right]do, \\ (119)_B\ \sum_{\omega,o}\varphi \equiv \iint\varphi f do d\omega, & (140)_B\ \frac{d}{dt}\sum_{\omega,o}\varphi = \iint\left(f\frac{\partial\varphi}{\partial t} + \varphi\frac{\partial f}{\partial t}\right)do d\omega = \sum_{n=1}^{5} C_n(\varphi) \end{cases}$$

5.5.1. **Conformation of $A_n(\varphi)$.**

$$\begin{cases} (121)_B & A_1(\varphi) = \frac{\partial\varphi}{\partial t}f, \\ (122)_B & A_2(\varphi) = -\varphi\left(\xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \zeta\frac{\partial f}{\partial z}\right), \\ (123)_B & A_3(\varphi) = -\varphi\left(X\frac{\partial f}{\partial x} + Y\frac{\partial f}{\partial y} + Z\frac{\partial f}{\partial z}\right), \\ (124)_B & A_4(\varphi) = \varphi\iint_0^\infty\int_0^{2\pi}(f'F_1' - fF_1)gb\, d\omega_1\, db\, d\epsilon, \\ (125)_B & A_5(\varphi) = \varphi\iint_0^\infty\int_0^{2\pi}(f'f_1' - ff_1)gb\, d\omega_1\, db\, d\epsilon, \end{cases}$$

where $\{A_n(\varphi)\}_{n=1}^{5}$ correspond to the effects such as

$$\begin{cases} A_1(\varphi): \text{the explicit dependence of } \varphi \text{ on } t; \\ A_2(\varphi): \text{the motion of the molecules;} \\ A_3(\varphi): \text{the external forces;} \\ A_4(\varphi): \textit{collisions} \text{ of } m\text{-molecules with } m_1\text{-molecules;} \\ A_5(\varphi): \textit{collisions} \text{ of } m\text{-molecules with each other;} \end{cases}$$

In order to find $\frac{\partial}{\partial t}\sum_{\omega,do}\varphi$, we have simply to integrate $\frac{\partial}{\partial t}\sum_{\omega,do}\varphi$ over all possible values of $d\omega$.

5.5.2. **Conformation of $B_n(\varphi)$.**

$$(126)_B\quad \frac{\partial}{\partial t}\sum_{\omega,do}\varphi = \left[\sum_{n=1}^{5} B_n(\varphi)\right]do.$$

One obtains each B by multiplying the corresponding A by $d\omega = d\xi d\eta d\zeta$ and integrating over all these variables from $-\infty$ to $+\infty$, which we indicate by a single integral sign. Thus :

$$(127)_B \quad B_1(\varphi) = \int A_1(\varphi)d\omega = \int \frac{\partial\varphi}{\partial t} f d\omega$$

$$(128)_B \quad B_2(\varphi) = \int A_2(\varphi)d\omega = -\int \varphi\Big(\xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \zeta\frac{\partial f}{\partial z}\Big)d\omega$$

$$(129)_B \quad B_3(\varphi) = \int A_3(\varphi)d\omega = -\int \varphi\Big(X\frac{\partial f}{\partial x} + Y\frac{\partial f}{\partial y} + Z\frac{\partial f}{\partial z}\Big)d\omega$$

$$(134)_B \quad B_4(\varphi) = \int A_4(\varphi)d\omega = \frac{1}{2}\iiint_0^\infty \int_0^{2\pi} (\varphi - \varphi')(f'F_1' - fF_1)gb\ d\omega\ d\omega_1\ db\ d\epsilon$$

$$(135)_B \quad B_{51}(\varphi) = \int A_5(\varphi' - \varphi)d\omega = \iiint_0^\infty \int_0^{2\pi} (\varphi' - \varphi)ff_1 gb\ d\omega\ d\omega_1\ db\ d\epsilon$$

$$(136)_B \quad = \iiint_0^\infty \int_0^{2\pi} (\varphi - \varphi')f'f_1' gb\ d\omega\ d\omega_1\ db\ d\epsilon$$

$$(135')_B \quad B_{51}'(\varphi) = \int A_5(\varphi_1' - \varphi_1)d\omega = \iiint_0^\infty \int_0^{2\pi} (\varphi_1' - \varphi_1)ff' gb\ d\omega\ d\omega_1\ db\ d\epsilon$$

$$(136')_B \quad B_{52}'(\varphi) = \int A_5(\varphi_1 - \varphi_1')d\omega = \iiint_0^\infty \int_0^{2\pi} (\varphi_1 - \varphi_1')f'f_1' gb\ d\omega\ d\omega_1\ db\ d\epsilon$$

From $(135)_B$,

$$(137)_B \quad B_{53}(\varphi) = \frac{1}{2}(B_{51} + B_{51}') = \frac{1}{2}\iiint_0^\infty \int_0^{2\pi} (\varphi' + \varphi_1' - \varphi - \varphi_1)ff' gb\ d\omega\ d\omega_1\ db\ d\epsilon$$

From $(136)_B$,

$$(138)_B \quad B_{54}(\varphi) = \frac{1}{2}(B_{52} + B_{52}') = \frac{1}{2}\iiint_0^\infty \int_0^{2\pi} (\varphi + \varphi_1 - \varphi' - \varphi_1')f'f_1' gb\ d\omega\ d\omega_1\ db\ d\epsilon$$

The arithmetic mean of $(137)_B$ and $(138)_B$,

$$(139)_B \quad B_5(\varphi) = \frac{1}{2}(B_{53} + B_{54}) = \frac{1}{4}\iiint_0^\infty \int_0^{2\pi} (\varphi + \varphi_1 - \varphi' - \varphi_1')(f'f_1' - ff_1)gb\ d\omega\ d\omega_1\ db\ d\epsilon$$

5.5.3. **Conformation of $C_n(\varphi)$.**

$$(140)_B \quad \frac{d}{dt}\sum_{\omega,o}\varphi = \sum_{n=1}^{5} C_n(\varphi) = \underbrace{C_1(\varphi) + C_2(\varphi) + C_3(\varphi)}_{\text{increments except for those resulting from collisions}} + \underbrace{C_4(\varphi) + C_5(\varphi)}_{\text{increments of those resulting from collisions}}$$

Remark: since in $\sum_{\omega,o}\varphi$ of $(140)_B$ one has to integrate over all values of do and $d\omega$, this quantity is now a function only of time. Hence the use of symbol $\frac{\partial}{\partial t}$ is unnecessary, and we can express differentiation by the usual Latin letter d. Each C is obtained by multiplying the corresponding B by do and integrating over all volume elements, or else by multiplying the coresponding A by $dod\omega$ and integrating over all do and $d\omega$ as we show in $(119)_B$.

Integrating $\{B_n(\varphi)\}_{n=1}^3$ of $(127)_B$, $(128)_B$, $(129)_B$ by do from $-\infty$ to $+\infty$,

$$(141)_B \quad C_1(\varphi) + C_2(\varphi) + C_3(\varphi) = \iint f dod\omega\Big(\frac{\partial\varphi}{\partial t} + \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \zeta\frac{\partial f}{\partial z} + X\frac{\partial f}{\partial x} + Y\frac{\partial f}{\partial y} + Z\frac{\partial f}{\partial z}\Big)$$

Integrating $B_4(\varphi)$ of $(134)_B$ by do from $-\infty$ to $+\infty$,

$$(142_1)_B \quad C_4(\varphi) = \frac{1}{2}\iiiint_0^\infty \int_0^{2\pi} (\varphi - \varphi')(f'F_1' - fF_1)gb\ do\ d\omega\ d\omega_1\ db\ d\epsilon$$

Integrating $B_5(\varphi)$ of $(139)_B$ by do from $-\infty$ to $+\infty$,

$$(142_2)_B \quad C_5(\varphi) = \frac{1}{4}\iiiint_0^\infty \int_0^{2\pi} (\varphi + \varphi_1 - \varphi' - \varphi_1')(f'f_1' - ff_1) gb\, do\, d\omega\, d\omega_1\, db\, d\epsilon$$

5.5.4. **More general proof of the entropy theorem. Treatment of the equations corresponding to the stationary state.** Boltzmann assert the following conditions

$$(147)_B \quad ff_1 = f'f_1', \quad FF_1 = F'F_1', \quad fF_1 = f'F_1'.$$

5.5.5. **Linearity of A_k, B_k, C_k.**

Since A, B, C are only the increments of definite quantities resulting from specified causes, most authors express them as derivatives of those quantities. Maxwell writes $\frac{\partial}{\partial t}\sum_{\omega,\, do}\varphi$, Kirchhoff $\frac{D}{Dt}\sum_{\omega,\, do}\varphi$ for $B_5(\varphi)$ etc. As with all differentials, the A for a sum of two functions is equal to the A's for the addends :

$$\begin{cases} A_k(\varphi+\psi) = A_k(\varphi) + A_k(\psi), \\ B_k(\varphi+\psi) = B_k(\varphi) + B_k(\psi), \\ C_k(\varphi+\psi) = C_k(\varphi) + C_k(\psi) \end{cases}$$

for any subscript k. These equations follows from the circumstance that φ occurs in all the integrals A, B, C only linearly.

5.6. **Special form of the incompressible, hydrodynamic equations.**

$$(171)_B \quad \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

$$(173)_B \quad \begin{cases} \rho\Big(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\Big) = \rho X - \frac{\partial(\rho\overline{\xi_0^2})}{\partial x} - \frac{\partial(\rho\overline{\xi_0\eta_0})}{\partial y} - \frac{\partial(\rho\overline{\xi_0\zeta_0})}{\partial z}, \\ \rho\Big(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\Big) = \rho Y - \frac{\partial(\rho\overline{\xi_0\eta_0})}{\partial x} - \frac{\partial(\rho\overline{\eta_0^2})}{\partial y} - \frac{\partial(\rho\overline{\zeta_0\eta_0})}{\partial z}, \\ \rho\Big(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\Big) = \rho Z - \frac{\partial(\rho\overline{\xi_0\zeta_0})}{\partial x} - \frac{\partial(\rho\overline{\eta_0\zeta_0})}{\partial y} - \frac{\partial(\rho\overline{\zeta_0^2})}{\partial z} \end{cases}$$

Boltzmann says "these equations as well as Equation $(171)_B$, are *only special cases of the general equation* $(126)_B$ and were derived from it by Maxwell and (following him) by Kirchhoff." Boltzmann concludes that if one collects all these terms, then Equation (126) reduces in this special case to:

$$(177)_B \quad \frac{\partial(\rho\overline{\varphi})}{\partial t} + \frac{\partial(\rho\overline{\xi\varphi})}{\partial x} + \frac{\partial(\rho\overline{\eta\varphi})}{\partial y} + \frac{\partial(\rho\overline{\zeta\varphi})}{\partial z} - \rho\Big[X\overline{\frac{\partial\varphi}{\partial\xi}} + Y\overline{\frac{\partial\varphi}{\partial\eta}} + Z\overline{\frac{\partial\varphi}{\partial\zeta}}\Big] = \underbrace{m\Big[B_4(\varphi) + B_5(\varphi)\Big]}_{\textit{collision terms}}$$

Boltzmann states about $(177)_B$:

From this equation Maxwell calculated the viscosity, diffusion, and heat conduction and Kirchhoff therefore calls it the basic equation of the theory. If one sets $\varphi = 1$, he obtains at once the continuity equation (171); for it follows from Equations (134) and (137) that $B_4(1) = B_5(1) = 0$. Subtraction of the continuity equation, multiplied by φ, from (177) gives (using the substitution [158]): [1, p.152].

where, (158) : $\xi = \xi_0 + u, \quad \eta = \eta_0 + v, \quad \zeta = \zeta_0 + w.$

$$(178)_B \quad \rho\Big(\frac{\partial\overline{\varphi}}{\partial t} + u\frac{\partial\overline{\varphi}}{\partial x} + v\frac{\partial\overline{\varphi}}{\partial y} + w\frac{\partial\overline{\varphi}}{\partial z}\Big) + \frac{\partial(\rho\overline{\xi_0\varphi})}{\partial x} + \frac{\partial(\rho\overline{\eta_0\varphi})}{\partial y} + \frac{\partial(\rho\overline{\zeta_0\varphi})}{\partial z} - \rho\Big[X\overline{\frac{\partial\varphi}{\partial\xi}} + Y\overline{\frac{\partial\varphi}{\partial\eta}} + Z\overline{\frac{\partial\varphi}{\partial\zeta}}\Big] = \underbrace{m\Big[B_4(\varphi) + B_5(\varphi)\Big]}_{\textit{collision terms}}$$

If one denotes the six quantities $(179)_B : \rho\overline{\xi_0^2}, \rho\overline{\eta_0^2}, \rho\overline{\zeta_0^2}, \rho\overline{\eta_0\zeta_0}, \rho\overline{\xi_0\zeta_0}, \rho\overline{\xi_0\eta_0}$ by X_x, Y_y, Z_z, $Y_z = Z_y$, $Z_x = X_z$, $X_y = Y_x$, namely, when we use the symmetric tensor, then we get the following :

$$\begin{bmatrix} \rho\overline{\xi_0^2} & \rho\overline{\xi_0\eta_0} & \rho\overline{\xi_0\zeta_0} \\ \rho\overline{\xi_0\eta_0} & \rho\overline{\eta_0^2} & \rho\overline{\eta_0\zeta_0} \\ \rho\overline{\xi_0\zeta_0} & \rho\overline{\zeta_0\eta_0} & \rho\overline{\zeta_0^2} \end{bmatrix} = \begin{bmatrix} X_x & X_y & X_z \\ Y_x & Y_y & Y_z \\ Z_x & Z_y & Z_z \end{bmatrix} = \begin{bmatrix} P_1 & T_3 & T_2 \\ T_3 & P_2 & T_1 \\ T_2 & T_1 & P_3 \end{bmatrix}, \qquad (26)$$

$$
(180)_B \quad \begin{cases} \rho\Big(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\Big)+\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}=\rho X, \\ \rho\Big(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\Big)+\frac{\partial Y_x}{\partial x}+\frac{\partial Y_y}{\partial y}+\frac{\partial Y_z}{\partial z}=\rho Y, \\ \rho\Big(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\Big)+\frac{\partial Z_x}{\partial x}+\frac{\partial Z_y}{\partial y}+\frac{\partial Z_z}{\partial z}=\rho Z \end{cases}
$$

These are not NS equations for lack of the pressure term. Moreover $(181)_B$: $p=\overline{\rho\xi_0^2}=\overline{\rho\eta_0^2}=\overline{\rho\zeta_0^2}$, $\overline{\xi_0\eta_0}=\overline{\xi_0\zeta_0}=\overline{\eta_0\zeta_0}=0$. Here, he assumes that from the supposition of isotropy and homogeneity, $p=\frac{1}{3}(X_x+Y_y+Z_z)$, which is the same as the principle by Saint-Venant or Stokes.

He deduces a special case of the hydrodynamic equations as follows:

> For the present, we assume as a fact of experience that in gases the normal pressure is always nearly equal in all directions, and that tangential elastic forces are very small, so that Equations (181) are approximately true. Substitution of the values given by this equation into Equation (173) yields:

$$
(183)_B \quad \begin{cases} \rho\Big(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\Big)+\frac{\partial p}{\partial x}-\rho X=0, \\ \rho\Big(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\Big)+\frac{\partial p}{\partial y}-\rho Y=0, \\ \rho\Big(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\Big)+\frac{\partial p}{\partial z}-\rho Z=0 \end{cases}
$$

which are the so-called Euler equations in incompressible condition of $(171)_B$.

$$
(185)_B \quad \begin{cases} \rho\frac{\partial u}{\partial t}+\frac{\partial(\overline{\rho\xi_0^2})}{\partial x}+\frac{\partial(\overline{\rho\xi_0\eta_0})}{\partial y}+\frac{\partial(\overline{\rho\xi_0\zeta_0})}{\partial z}-\rho X=0, \\ \rho\frac{\partial v}{\partial t}+\frac{\partial(\overline{\rho\xi_0\eta_0})}{\partial x}+\frac{\partial(\overline{\rho\eta_0^2})}{\partial y}+\frac{\partial(\overline{\rho\eta_0\zeta_0})}{\partial z}-\rho Y=0, \\ \rho\frac{\partial w}{\partial t}+\frac{\partial(\overline{\rho\xi_0\zeta_0})}{\partial x}+\frac{\partial(\overline{\rho\zeta_0\eta_0})}{\partial y}+\frac{\partial(\overline{\rho\zeta_0^2})}{\partial z}-\rho Z=0 \end{cases}
$$

We set the values of (26) as follows, which is the same tensor as Stokes :

$$
(220)_B \quad \begin{bmatrix} \overline{\rho\xi_0^2} & \overline{\rho\xi_0\eta_0} & \overline{\rho\xi_0\zeta_0} \\ \overline{\rho\xi_0\eta_0} & \overline{\rho\eta_0^2} & \overline{\rho\eta_0\zeta_0} \\ \overline{\rho\xi_0\zeta_0} & \overline{\rho\zeta_0\eta_0} & \overline{\rho\zeta_0^2} \end{bmatrix} = \begin{bmatrix} p-2\mathcal{R}\Big\{\frac{\partial u}{\partial x}-\frac{1}{3}\Big(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\Big)\Big\} & -\mathcal{R}\Big(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\Big) & -\mathcal{R}\Big(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\Big) \\ -\mathcal{R}\Big(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\Big) & p-2\mathcal{R}\Big\{\frac{\partial v}{\partial y}-\frac{1}{3}\Big(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\Big)\Big\} & -\mathcal{R}\Big(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\Big) \\ -\mathcal{R}\Big(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\Big) & -\mathcal{R}\Big(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\Big) & p-2\mathcal{R}\Big\{\frac{\partial w}{\partial z}-\frac{1}{3}\Big(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\Big)\Big\} \end{bmatrix}
$$

From $(220)_B$, we calculate the components of $(185)_B$ as follows:

$$
\begin{bmatrix} \frac{\partial(\overline{\rho\xi_0^2})}{\partial x} & \frac{\partial(\overline{\rho\xi_0\eta_0})}{\partial y} & \frac{\partial(\overline{\rho\xi_0\zeta_0})}{\partial z} \\ \frac{\partial(\overline{\rho\xi_0\eta_0})}{\partial x} & \frac{\partial(\overline{\rho\eta_0^2})}{\partial y} & \frac{\partial(\overline{\rho\eta_0\zeta_0})}{\partial z} \\ \frac{\partial(\overline{\rho\xi_0\zeta_0})}{\partial x} & \frac{\partial(\overline{\rho\zeta_0\eta_0})}{\partial y} & \frac{\partial(\overline{\rho\zeta_0^2})}{\partial z} \end{bmatrix}
= \begin{bmatrix} p-\mathcal{R}\Big\{2\frac{\partial u}{\partial x}-\frac{2}{3}\Big(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\Big)\Big\} & -\mathcal{R}\Big(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\Big) & -\mathcal{R}\Big(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\Big) \\ -\mathcal{R}\Big(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\Big) & p-\mathcal{R}\Big\{2\frac{\partial v}{\partial y}-\frac{2}{3}\Big(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\Big)\Big\} & -\mathcal{R}\Big(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\Big) \\ -\mathcal{R}\Big(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\Big) & -\mathcal{R}\Big(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\Big) & p-\mathcal{R}\Big\{2\frac{\partial w}{\partial z}-\frac{2}{3}\Big(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\Big)\Big\} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}
$$

Then, substitution of these values into the equations of motion $(185)_B$ yields:

$$
(221)_B \quad \begin{cases} \rho\frac{\partial u}{\partial t}+\frac{\partial p}{\partial x}-\mathcal{R}\Big[\Delta u+\frac{1}{3}\frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\Big)\Big]-\rho X=0, \\ \rho\frac{\partial v}{\partial t}+\frac{\partial p}{\partial y}-\mathcal{R}\Big[\Delta v+\frac{1}{3}\frac{\partial}{\partial y}\Big(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\Big)\Big]-\rho Y=0, \\ \rho\frac{\partial u}{\partial t}+\frac{\partial p}{\partial z}-\mathcal{R}\Big[\Delta w+\frac{1}{3}\frac{\partial}{\partial z}\Big(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\Big)\Big]-\rho Z=0 \end{cases}
$$

We can interpret that as the special cases, Boltzmann have deduced the NS equations after substituting the tensor $(220)_B$ to $(173)_B$, for lack of pressure terms.

We can construct the tensor with the Equations (13) and (14) as follows:

$$\begin{bmatrix} \rho\xi^2 & \rho\xi\eta & \rho\xi\zeta \\ \rho\xi\eta & \rho\eta^2 & \rho\eta\zeta \\ \rho\xi\zeta & \rho\zeta\eta & \rho\zeta^2 \end{bmatrix} = \begin{bmatrix} p - \frac{M}{9k\rho\Theta_2}p\left(2\frac{du}{dx} - \frac{dv}{dy} - \frac{dw}{dz}\right) & -\frac{M}{6k\rho\Theta_2}p\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & -\frac{M}{6k\rho\Theta_2}p\left(\frac{dw}{dx} + \frac{du}{dz}\right) \\ -\frac{M}{6k\rho\Theta_2}p\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & p - \frac{M}{9k\rho\Theta_2}p\left(\frac{du}{dx} - 2\frac{dv}{dy} - \frac{dw}{dz}\right) & -\frac{M}{6k\rho\Theta_2}p\left(\frac{dv}{dz} + \frac{dw}{dy}\right) \\ -\frac{M}{6k\rho\Theta_2}p\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & -\frac{M}{6k\rho\Theta_2}p\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & p - \frac{M}{9k\rho\Theta_2}p\left(\frac{du}{dx} - \frac{dv}{dy} - 2\frac{dw}{dz}\right) \end{bmatrix} \quad (27)$$

From $\mathcal{R} \equiv \frac{M}{6k\rho\Theta_2}p$, we get $(220)_B$. The equations (11) equals $(185)_B$ and (12) equals $(221)_B$ except for the coefficient.

5.7. **Entropy.**

The word entropy was deduced by Clausius [7] in 1865, and following his nomenclature, Boltzmann constructed his first version of equations in 1872, applying entropy to his gas theory. We show citing [7] Clausius' Greek nomenclature, meaning "conversion" of material as follows :

$$(60)_C \quad S = S_0 + \int \frac{dQ}{T}, \quad (65)_C \quad \int \frac{dQ}{T} = S - S_0,$$

welch, nur etwas anders geordnet, dieselb ist, wie die unter (60) angeführt zur Bestimung von S dienendende Gleichung.

Sucht man für S einen bezeichnenden Namen, so könnte man, ähnlich wie von der Grösse U gesagt ist, sie sey der *Wärme - und Wirkinhalt* des Körpers. Das ich es aber für besser halt, die Namen derartiger für dir Wissenschaft wichtiger Grössen aus den alten Sprachen zu entnehmen, damit sie unverändert in allen neuen Sprechen angewandt werden können, so schlage ich vor, die Grösse S nach dem griechischen Worte ἡ τροπὴ, die Verwandlung, die *Entropie* des Körpers zu nennen.

Das Wort *Entropie* habe ich absichtlich dem Wort *Entropie* durch diese Worte banannte werden sollen, sind ihren physikalischen Bedeutung nach einander so nahe verwandt, dass eine gewisse Gleichartigkeit in der Benennung mir zweckmässig zu seyn scheint. [7, 389-390]

(Transl.) $(60)_C$, $(65)_C$, which seemed to be like only reallocated expression, however, the usage cited in $(60)_C$, is useful equation.

We sought some suitable name for the nomenclature for S, like the quantity U, such as the value of warm and value of work of a material. I considered that it seemed to be suitable to be adopted from the old Greek as the nomenclature for the important quantity, so I owed it to the quantity S from Greek word ἡ τροπὴ, which means "conversion", the *Entropy* of the material. ⋯

Boltzmann consider when the following conditions do not hold, where, the number of the two molecules f and f_1, F and F_1 and f and F_1 before and after collision, namely from $(147)_B$,

$$ff_1 \neq f'f_1', \quad FF_1 \neq F'F_1', \quad fF_1 \neq f'F_1'.$$

We construct the expression H for the gas contained in the volume element do. The value thus found will be multiplied by $-RM$ and divided by do. Let this quantity be

$$J = -RM \int f \ln f d\omega.$$

Jdo is then the "entropy" of the gas contained in do, if it had the same energy (heat) content and the same progressive motion in space, and obeyed the Maxwell velocity distribution law. It can be calculated just as in §19, and has the value

$$\frac{R\rho}{\mu} \ln(\frac{T^{\frac{3}{2}}}{\rho})$$

here, this value $\frac{R\rho}{\mu}$ is called *Boltzmann constant* and it was inscribed on his epitaph as

$$S = k \ln w$$

which is also

$$(\frac{T^{\frac{3}{2}}}{\rho})^k = \exp S$$

6. Conclusions. Contributions to the NS equations

Basically, the NS equations were deduced from Newton's kinetic equation (the second law of motion) : $\mathbf{F} = m\mathbf{r}$, [22] however Boltzmann's gas equations were not deduced from it, but he extended the ideas of gas theory including the problem of gas collision by its progenitors Maxwell and Kirchhoff. In fact, Boltzmann had confessed his fear the authority in the preface of the Part II of his book (cf. Appendix).

When we consider the contributions by Boltzmann to the NS equations, Boltzmann shows the Euler equations and the NS equation as the special case of his general hydrodynamic equations. He verified the validity of the Euler equations and the NS equations, which were recognized in 1934 at latest by Prandtl [39, p.259], and at the epoch about one hundred years after Navier's paper [32], read by the referees in 1822 and published in *Mémoires de L'Academie des Sience de l'Institute de France* in 1827.

Maxwell in 1865, Boltzmann in 1895 and Prandtl[38, 39] in 1904 both used the "well-known hydrodynamic equations" and at latest in 1929, used the nomenclature of "Navier-Stokes equations", using the two-constant not of Navier, but of Saint-Venant, Stokes, and expanded by Maxwell, Kirchhoff and Boltzmann. These three persons verified the hydrodynamic equations without the name as Navier-Stokes equations.

In short, we can state that after formulating by Navier (1827) [32], Cauchy (1828) [5], Poisson (1831) [35], Saint-Venant (1843) [41] and Stokes (1849) [42], the topics of hydrodynamic history are rebuilt by Maxwell (1865) [29], Boltzmann (1895) [1] and Prandtl (1927) [39] in the cyclic interval of about 30 years or so.

As the two constants, Saint-Venant had used ε and $\frac{\varepsilon}{3}$, and Stokes μ and $\frac{\mu}{3}$, while Boltzmann used $\mathcal{R}$ and $\frac{\mathcal{R}}{3}$ after tracing Maxwell. According to Prandtl[38], we can suppose that the naming may be decided in "The third international mathematical Congress" in Heidelberg in 1904 or few years later than it. Boltzmann states hydrodynamic equations as well as the Euler equations of $(183)_B$:

> Die Gleichungen 221 sind die bekannten auf innere Reibung corrigirten hydrodynamischen Gleichungen. [2, p.169]
>
> (transl.) Equations (221) are the well-known hydrodynamic equations corrected for internal viscosity. [1, p.176]

According to Boltzmann's description, we can suppose the fact that the then academic society had not fixed yet the name of this equations, up to 1895 or 1898.

7. Epilogue. Humanity of Boltzmann

In 1898, Boltzmann had published *Vorlesungen über Gastheorie*, II Teil. (*The lecture of gas theory,* Part II), in which preface, he had expressed his fear that the theory of gases were temporarily thrown into oblibion as follows :

> Es wäre daher meines Erachtens ein Schaden für die Wissenschaft, wenn die Gastheorie durch die augenblicklich herrschende ihr feindselige Stimmung zeitweilig in Vergessenheit geriethe, wie z.B. einst die Undulationstheorie durch die Autorität Newton's. [2, Vorwort]
>
> In my opinion it would be a great tragedy for science if the theory of gases were temporarily thrown into oblibion because of a momentary hostile attitude toward it, as was for example the wave theory because of Newton's authority. Forward to Part II. [1, p.215]

After eight years, a newspaper in Wien '*Neue Freie Presse*', (New Free Press, Wien, Freitag, 07/September in 1906, Nr. 15102) reports Mach's consternation confronted by the news of Boltzmann who had taken his life. Here we cite our transcription from the Fraktur printing style of the newspaper in

[22](⇓) By d'Alembert's principle in 1758, from the Newton's kinetic equation (the second law of motion) : $\mathbf{F} = m\mathbf{r}$, d'Alembert proposed $\mathbf{F} - m\mathbf{r} = 0$, where, $\mathbf{F}$: the force, m : the gravity, $\mathbf{r}$: the acceleration. According to his assertion, the problem of kinetic dynamics turns into that of the static dynamics.

1906, which is in Broda [3] [23], and we show it in our last page of our paper, thanking Saburo Ichii and Toshihiko Tsuneto and the publishing company Misuzu Shobo. From here, we can see Boltzmann was having both the ardent passion to the learning and the pure humanity in his lifetime.

Remark. Mach had been the supervisor of Boltzmann and both were the then position of 'Hofrat', namely the advisor to Court of the Empire of Austria-Hungary, [24] so that the news reads 'Hofrat Mach' or 'Hofrat Boltzmann'.

Hofrat Professor Mach über den Tod Boltzmanns.

Hofrat. Mach, der durch den Tod Boltzmanns zehr schmerzlich berührt worden ist, feilte und mit, daß das fraurige Ende der durch Selbstmord geräde jetzt nicht zu befürchten war, da sich sein geistiger Zustand in der lasten Zeit etwas gebessert hatte. Seit etwas zwei Jahren war zu er allerdings Unfällen von Irrwahn ausgefahrt, in denen sich bei ihm namentlich der Trieb zur Flucht fühlbar machte. Er mußte deshalb sorgfältig über macht werden. Doch traten wieder Momente ein, in denen er beruhigender Zusprache zugänglich war. Dies war auch der Fall, als er zur Erholung nach Duino gebracht wurde. Er versprach sich ruhig zu verhalten, und die Familie glaubte, daß die Besserung anhalten werde, so daß man nicht aus den Eintritt seiner verbürgten Gerüchten zufolge hat Boltzmann schon damals verführt, Hand an sich zu legen.

Gelegentlich der Unwesenheit von Professor Dftmalb in Wien habe ich Boltzmann zum leztenmal in wirtlich froher Laune gesehen, in so guter Stimmung, wie selten vorher und nie wieder seither. Wir wohnten damals zusammen den Borirägen des Berliner Gastes im Ingenieur- und Architektenverein bei und zum Abschied war die Sachwelt bei einem Bankett vereinigt. Dftmalb saß auf den Ehrenplatz, Boltzmann zu seiner Rechten und ich zur Linken. Die "Glücksformel", die Dftmalb entwickelt hatte, gab Boltzmann Anlaß zu einer geistsprühenden den Tischrede. Lange saßen wir beisammen, und nach Mittelnacht geleitete ich ihn heim. Boltzmann war von einer kindlichen Reinheit des Geistes, von unerschöpflicher Liebenswürdigkeit und glücklich, wenn er jemanden gefällig sein konnte.

Un Unerkennung als Gelehrter hat es ihm nie geschkt. Seine Bedeutung war je überagend, daß man sich ihr nicht entziehen konnt. Es war ihm auch beschieden, aus dem Kreise seiner Schüler große Männer hervorgehen zu sehen. Der Schwede Arrhenius, der Berliner Bernst, beide Koryphäen der Wissenschaft, waren Hörer Boltzmanns, und beide haben oft betont, wie unendlich viel sie ihrem Meister zu danken haben. Nach der Pensionierung von Professor Mach hat Hofrat Boltzmann auch philosophische Vorfräge gehalten, die sich außerordentlich guten Besuches zu erfreuen hatten.

Es ist ein Jammer, daß ein Mensch von der gewartigen Bedeutung Boltzmanns vor der Zeit aus dem Leben geschieden ist. Er hat der Wissenscaft Immenses geleistet, aber es war immer noch Prozeß von ihm zu erhoffen.

Translated sketches of the news story :

Mach was surprised at the news of Boltzmann's death. Mach had heard that Boltzmann was saying himself his recent steady calm, so all the members of the family had supposed that Boltzmann was recovering from being in the low spirits and had not been afraid of such an imminent state of mind.

We lived then together with the gests from Berlin of the association of tecknology and architecture in Borirägen. He avoided the drinking party or banquet for his standard of value.

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[23](⇓) The original by Broda didn't cite this newspaper, however, the translators into Japanese [3] cites a photo of the then news stories in the Fraktur printing style. Here we cite our transcription from the Fraktur printing style into the today's German style for convenience' sake.

[24](⇓) The Empire of Austria-Hungary : 1867-1918.

Dftmalb took the seat of honor, to whom Boltzmann sat the right side and I the left side. Dftmalb proposes "the formula of happiness", Boltzmann gave the oppotunities for the speech. We were sitting together with him. At midnight, we went back to home.

Boltzmann had a childish unalloyed genuine of mind and devoted endless kindness in perfect happyness to anybody, whom, when he could be kind to.

His temperate obstinancy as a scholer didn't allow him to play his cards well. His idea was so noble that one should have not been easy to get along with him. Boltzmann kept away from the troubles with the scholars.

Arrhenius of Swedish and Bernst of the Berliner were the authorities in each academic arena and colaborators of studies with Boltzmann and also the good listeners of Boltzmann's talks, and both have emphasized that how very frequently they had thanked their savant, Boltzmann. Boltzmann gave also the lectures on philosophy.

The interviewee, Mach concludes his talk in the last paragraph with the following evaluation to Boltzmann : "It is greatly to be regretted that a promissing person upon his future, considering the importance of Boltzmann, passed away his life. He had achieved the great tasks, however, it was still under the process of extending it eternally."

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Remark: we use *Lu* (: in French) in the bibliography meaning "read" date by the referees of the journals, for example MAS. In citing the original paragraphs in our paper, the underscoring are of ours.

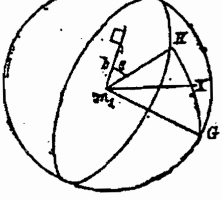

Fig. 6.

Hofrat Professor Mach über den Tod Boltzmanns.

Hofrat Mach, der durch den Tod Boltzmanns sehr schmerzlich berührt worden ist, teilte uns mit, daß das traurige Ende des Gelehrten durch Selbstmord gerade jetzt nicht zu befürchten war, da sich sein geistiger Zustand in der letzten Zeit etwas gebessert hatte. Seit etwa zwei Jahren war er allerdings Anfällen von Irrwahn ausgesetzt, in denen sich bei ihm namentlich der Trieb zur Flucht fühlbar machte. Er mußte deshalb sorgfältig überwacht werden. Doch traten wieder Momente ein, in denen er beruhigender Zusprache zugänglich war. Dies war auch der Fall, als er zur Erholung nach Duino gebracht wurde. Er versprach, sich ruhig zu verhalten, und die Familie glaubte, daß die Besserung anhalten werde, so daß man nicht auf den Eintritt einer verbürgten Gerüchten zufolge hat Boltzmann schon damals versucht, Hand an sich zu legen

Gelegentlich der Anwesenheit von Professor Ostwald in Wien habe ich Boltzmann zum letztenmal in wirklich froher Laune gesehen, in so guter Stimmung, wie selten vorher und nie wieder seither. Wir wohnten damals zusammen den Vorträgen des Berliner Gastes im Ingenieur- und Architektenverein bei und zum Abschied war die Fachwelt bei einem Bankett vereinigt. Ostwald saß auf dem Ehrenplatz, Boltzmann zu seiner Rechten und ich zur Linken. Die „Glücksformel", die Ostwald entwickelt hatte, gab Boltzmann Anlaß zu einer geistsprühenden Tischrede. Lange saßen wir beisammen, und nach Mitternacht geleitete ich ihn heim. Boltzmann war von einer kindlichen Reinheit des Geistes, von unerschöpflicher Liebenswürdigkeit und glücklich, wenn er jemandem gefällig sein konnte.

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