EXPLICIT CONSTRUCTIONS OF CASIMIR OPERATORS OF sl(n;C) AND so(n;R)

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1. Historical introduction

In 1931, Hendrik Brugt Gerhard Casimir (1909-2000) found out the foremost important quadratic sum (i.e. second-order Casimir operator) of elements in a Lie algebra. Then he and van der Waerden used it for a proof of completely reducibility of the representations of a semisimple Lie algebra.

This Casimir operator was also used for a proof of Levi decomposition theorem on a finite-dimensional Lie algebra over a field with characteristic zero. An algebraic proof, which means to use neither Lie group nor analytic method, of the Weyl character formula on the irreducible representation with highest weight of a semisimple Lie algebra has been accomplished by H.Freudenthal via Casimir operator chasing.

Although Harish-Chandra enunciated the center of the universal enveloping algebra of a Lie algebra via Cartan-Weyl theory, explicit construction of the generators of its center has been carried out by G.Racah around 1951, by introducing the higher-order generalized Casimir operators.

Then cohomological theory of Lie algebras has showed up through geometric treatments. So-called exponents of simple Lie algebras are related to the degrees of the generators of the center of their universal enveloping algebras.

2. Casimir operator

Let g be a r-dimensional semisimple Lie algebra over C, and let $\{u_1, \ldots, u_r\}$ be a basis of g. For a n-dimensional faithful representation $p : g \rightarrow gl(n;C) = Mat(n;C)$, we write by $U_i = p(u_i)$ for brevity. Since $g_{ij} = B_p(u_i, u_j) = Tr(U_i U_j)$ becomes a non-degenerate symmetric bilinear form, there exists the inverse matrix (g^{ij}) of (g_{ij}). By introducing $U^j = \sum g^{ij} U_i$, we have the following equations ;

$$\sum g^{ij} U_i U_j = \sum U^j U_j = \sum g_{ij} U^i U^j = \sum U_j U^j \quad (\,;\ \text{say } C\,).$$

Then we obtain the dual basis $\{u^1, u^2, \ldots, u^r\}$ of $\{u_1, \ldots, u_r\}$ with respect to trace-form B_p such that $B_p(u^i, u_j) = \delta_{ij}$ and $p(u^i) = U^i$ $(1 \leqq i \leqq r)$. The above matrix C is called the Casimir operator of $(g\,;\,p)$.

Let $U(g)$ be the universal enveloping algebra of g , and let $Z(U(g))$ be the center of $U(g)$. The element $c = \sum u^j u_j$ is called the Casimir element of $(g ; p)$. It is known that $C = (r/n) I_n = (\dim g / \dim p) I_n$ and $c = \sum u^j u_j \in Z(U(g))$.

Proposition 1. The Casimir operator does not depend on the choice of basis.

Proof. Suppose that $\{ v_i \}$ is another basis of g . Write by $v_i = \sum a_{ij} u_j$, then there exists the inverse matrix $A^{-1} = (a^{ij})$ of $A = (a_{ij})$. We define $h_{ij} = B_p (v_i , v_j) = \sum a_{ik} g_{kl} a_{jl}$. Since g is semisimple, there exists the inverse matrix (h^{ij}) of (h_{ij}) . It follows from $(h_{ij}) = A (g_{ij}) {}^t A$ that $(h^{ij}) = ({}^t A)^{-1} (g^{ij}) A^{-1}$.

Hence $\sum h^{ij} V_i V_j = \sum a^{ki} g^{kl} a^{lj} V_i V_j = \sum g^{kl} U_k U_l = C$.

This proves our claim. Q.E.D.

3. Basic example

The origin of Casimir operator may likely be three-dimensional simple Lie algebra sl(2;C) . Since $sl(2;C) = \{ X \in Mat(2;C) \ ; \ Tr(X) = 0 \}$ and $B(x,y) = Tr(XY)$ is non-degenerate bilinear form, one sees that $\{ f, e, h/2 \}$ is the dual basis of $\{ e, f, h \}$, where

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Then there are several ways of calculation of the Casimir operator as follows.

$$C = Tr(ee)ff + Tr(ef)fe + Tr(eh)fh/2 + Tr(fe)ef + Tr(ff)ee$$

$$+ Tr(fh)eh/2 + Tr(he)h/2f + Tr(hf)h/2\, e + Tr(hh)\, h/2\, h/2$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2 \begin{pmatrix} 1/4 & 0 \\ 0 & 1/4 \end{pmatrix} = 3/2\ I_2 ,$$

$$C = e\,f + f\,e + h\ h/2 = 3/2\ I_2 .$$

4. Main theorem

Let p be identical injective representation of $sl(n;C)$, $so(n;R)$, respectively . In this section, we explicitly construct Casimir operator of $(sl(n;C) ; p)$, $(so(n;R) ; p)$, respectively.

(I) Let r be $(n^2 - n)/2$, and consider the following standard basis of $sl(n;C)$;

$e_1 = E_{12}$, $e_2 = E_{13}$, ... , $e_{n-1} = E_{1n}$, $e_n = E_{23}$, $e_{n+1} = E_{24}$,

... , $e_{r-1} = E_{n-2,n}$, $e_r = E_{n-1,n}$, $f_1 = E_{21}$, $f_2 = E_{31}$, ... ,

$f_{n-1} = E_{n1}$, $f_n = E_{32}$, $f_{n+1} = E_{42}$, ... , $f_{r-1} = E_{n,n-2}$, $f_r = E_{n,n-1}$,

$h_1 = E_{11} - E_{22}$, $h_2 = E_{22} - E_{33}$, ... , $h_{n-1} = E_{n-1,n-1} - E_{nn}$, where

E_{ij} denote matrix units, and $2r + (n-1) = n^2 - 1 = \dim (sl(n;C))$.

Now let us find out the dual basis with respect to Trace form $B_p(X,Y) = Tr(XY)$.

We define $n-1$ elements k_1 , k_2 , ... , k_{n-1} as follows.

$$k_1 = (n-1)/n \ E_{11} + (-1/n) \ E_{22} + (2-n)/n \ E_{33} ,$$

$$k_2 = (n-2)/n \ E_{22} + (-2/n) \ E_{33} + (4-n)/n \ E_{44} ,$$

$$k_3 = (n-3)/n \ E_{33} + (-3/n) \ E_{44} + (6-n)/n \ E_{55} ,$$

$$k_4 = (n-4)/n \ E_{44} + (-4/n) \ E_{55} + (8-n)/n \ E_{66} ,$$

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$$k_{n-3} = (n-(n-3))/n \ E_{n-3,n-3} + (-(n-3))/n \ E_{n-2,n-2} + (2(n-3)-n)/n \ E_{n-1,n-1}$$

$$k_{n-2} = (n-(n-2))/n \ E_{n-2,n-2} + (-(n-2))/n \ E_{n-1,n-1} + (2(n-2)-n)/n \ E_{nn}$$

$$k_{n-1} = (2(n-1)-n)/n \ E_{11} + (n-(n-1))/n \ E_{n-1,n-1} + (-(n-1))/n \ E_{nn}$$

Then $\{ f_1 , f_2 , \ldots , f_r , e_1 , e_2 , \ldots , e_r , k_1 , k_2 , \ldots , k_{n-1} \}$

is the dual basis of $\{ e_1 , \ldots , e_r , f_1 , \ldots , f_r , h_1 , \ldots , h_{n-1} \}$

such that $B_p(u^j , u_i) = \delta_{ij}$. It follows from $\sum h_i k_i = (n-1)/n \ I_n$

that $C = \sum e_i f_i + \sum f_i e_i + \sum h_i k_i = (n-1) I_n + (n-1)/n \ I_n$

$= (n^2 - 1) /n \ I_n = (\dim g) / (\dim p) \ I_n$.

(II) Let $\{ X_1, \ldots, X_r \}$ be a standard basis of $so(n;R)$

$= \{ X \in gl(n;R) = Mat(n;R) \; ; \; {}^tX + X = 0_n \}$ as follows.

$X_1 = E_{12} - E_{21}, \quad X_2 = E_{13} - E_{31}, \ldots, \quad X_n = E_{23} - E_{32},$

$\ldots, \quad X_r = E_{n-1,n} - E_{n,n-1}$, where $r = (n^2 - n)/2$

$= \dim(so(n;R))$.

Then one sees that $\{ Y_1, \ldots, Y_r \}$ is the dual basis of $\{ X_1, \ldots, X_r \}$

with respect to Trace form, here $Y_j = (-1/2) X_j \quad (1 \leqq j \leqq r)$. Thus we

know that $C = \sum X_j Y_j = (n-1)/2 \; I_n = ((n^2 - n)/2)/n \; I_n$

$= (\dim g) / (\dim p) \; I_n$.

5. Generalization

Let $\{u_i\}$ be a basis of a r-dimensional semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, and let $\rho : \mathfrak{g} \to \mathfrak{gl}(n;\mathbb{C}) = Mat(n;\mathbb{C})$ be a faithful representation of $\mathfrak{g}$. For brevity, we write by $U = \rho(u)$ $(u \in \mathfrak{g} = \oplus \mathbb{C}u_i)$. Let $B_\rho(u,v) = Tr(UV)$ be non-degenerate symmetric bilinear trace form of $(\mathfrak{g},\rho)$, and let $g_{ij} = B_\rho(u_i, u_j)$. Since (g_{ij}) is nonsingular, there exists the inverse matrix (g^{ij}). Put $u^i = \sum g^{ij} u_j$ $(1 \leqq i \leqq r)$, then one sees that $\{u^i\}$ is the dual basis of $\{u_j\}$ such that $Tr(U^i U_j) = \delta_{ij}$.

In 1951, G. Racah defined higher-order Casimir operators (i.e. generalized Casimir operator of order $t \geqq 2$) as follows.

$$C_t = \sum Tr(U_{i_1} U_{i_2} \cdots U_{i_t}) U^{i_1} U^{i_2} \cdots U^{i_t}$$

Furthermore, he constructed a complete set of generators of the center of the universal enveloping algebra of each simple Lie algebra.

Proposition 2. Under the above notations, C_t does not depend on the choice of basis $\{u_i\}$ of $\mathfrak{g}$.

Proof. The former of the proof of proposition 1 in section 2 is available for our proof with the same notations. Let $\{v_i\}$ be another basis such that $v_i = \sum a_{ij} u_j$. Write by $(a^{ij}) = (a_{ij})^{-1}$, $(h^{ij}) = (h_{ij})^{-1}$, where $h_{ij} = B_\rho(v_i, v_j) = Tr(V_i V_j) = \sum a_{il} a_{jm} g_{lm}$.

It follows from $(h^{ij}) = {}^t(a_{ij})^{-1} (g^{ij}) (a_{ij})^{-1}$ that

$$\sum a_{lj} V^l = \sum a_{lj} h^{lm} V_m = \sum a_{lj} h^{lm} a_{ms} U_s = \sum g^{js} U_s = U^j.$$

Hence $\sum Tr(V_{i_1} V_{i_2} \cdots V_{i_t}) V^{i_1} V^{i_2} \cdots V^{i_t}$

$$= \sum Tr(a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_t j_t} U_{j_1} U_{j_2} \cdots U_{j_t}) V^{i_1} V^{i_2} \cdots V^{i_t}$$

$$= \sum Tr(U_{j_1} U_{j_2} \cdots U_{j_t}) a_{i_1 j_1} V^{i_1} a_{i_2 j_2} V^{i_2} \cdots a_{i_t j_t} V^{i_t}$$

$$= \sum Tr(U_{j_1} U_{j_2} \cdots U_{j_t}) U^{j_1} U^{j_2} \cdots U^{j_t} = C_t .$$

This completes our proof. Q.E.D.

Proposition 3. Let $U(\mathfrak{g})$ be the universal enveloping algebra of a semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, and put $Z(U(\mathfrak{g}))$ its center. Then we have $C_t \in Z(U(\mathfrak{g}))$ for every integer $t \geqq 2$.

Proof. It is enough to prove that $C_3 \in Z(U(\mathfrak{g}))$ because our argument of C_3 also works for every integer $t \geqq 3$.

Recall the following coefficients $d_{i_j l}(u_k)$, $C_{i_j m}(u_k)$ $(1 \leqq k \leqq r,\ 1 \leqq l, m \leqq r = \dim \mathfrak{g},\ 1 \leqq j \leqq t)$;

$$[U_k, U^{i_j}] = \sum d_{i_j l}(u_k) U^l, \quad [U_k, U_{i_j}] = \sum C_{i_j m}(u_k) U_m .$$

Since $B_\rho([x,y],z) = B_\rho(x,[y,z])$, we have $d_{ij}(x) = B_\rho([x,u^i],u_j) = -B_\rho([u^i,x],u_j) = -C_{ji}(x)$ and $d_{i_j l}(u_k) = -C_{mi_j}(u_k)$.

Now let us consider $[U_k, C_3] = U_k C_3 - C_3 U_k$

$$= \sum \mathrm{Tr}(U_{i_1}U_{i_2}U_{i_3}) U_k U^{i_1}U^{i_2}U^{i_3} - \sum \mathrm{Tr}(U_{i_1}U_{i_2}U_{i_3}) U^{i_1}U^{i_2}U^{i_3}U_k$$

$$= \sum \mathrm{Tr}(U_{i_1}U_{i_2}U_{i_3}) U_k U^{i_1}U^{i_2}U^{i_3} - \sum \mathrm{Tr}(U_{i_1}U_{i_2}U_{i_3}) U^{i_1}U_k U^{i_2}U^{i_3}$$

$$+ \sum \mathrm{Tr}(U_{i_1}U_{i_2}U_{i_3}) U^{i_1}U_k U^{i_2}U^{i_3} - \sum \mathrm{Tr}(U_{i_1}U_{i_2}U_{i_3}) U^{i_1}U^{i_2}U_k U^{i_3}$$

$$+ \sum \mathrm{Tr}(U_{i_1}U_{i_2}U_{i_3}) U^{i_1}U^{i_2}U_k U^{i_3} - \sum \mathrm{Tr}(U_{i_1}U_{i_2}U_{i_3}) U^{i_1}U^{i_2}U^{i_3}U_k$$

$$
\begin{aligned}
&= \sum Tr(U_{i_1}U_{i_2}U_{i_3})\,[U_k, U^{i_1}]\,U^{i_2}U^{i_3} + \sum Tr(U_{i_1}U_{i_2}U_{i_3})\,U^{i_1}[U_k, U^{i_2}]\,U^{i_3} \\
&\quad + \sum Tr(U_{i_1}U_{i_2}U_{i_3})\,U^{i_1}U^{i_2}[U_k, U^{i_3}] \\
&= \sum Tr(U_{i_1}U_{i_2}U_{i_3})\, d_{i_1 \ell}(u_k)\, U^{\ell}U^{i_2}U^{i_3} \\
&\quad + \sum Tr(U_{i_1}U_{i_2}U_{i_3})\, U^{i_1}\, d_{i_2 \ell}(u_k)\, U^{\ell}U^{i_3} \\
&\qquad + \sum Tr(U_{i_1}U_{i_2}U_{i_3})\, U^{i_1}U^{i_2}\, d_{i_3 \ell}(u_k)\, U^{\ell} \\
&= \sum Tr\big(d_{i_1 \ell}(u_k)\, U_{i_1}U_{i_2}U_{i_3}\big)\, U^{\ell}U^{i_2}U^{i_3} \\
&\quad + \sum Tr\big(U_{i_1}\, d_{i_2 \ell}(u_k)\, U_{i_2}U_{i_3}\big)\, U^{i_1}U^{\ell}U^{i_3} \\
&\qquad + \sum Tr\big(U_{i_1}U_{i_2}\, d_{i_3 \ell}(u_k)\, U_{i_3}\big)\, U^{i_1}U^{i_2}U^{\ell} \\
&= \sum Tr\big(-C_{\ell i_1}(u_k)\, U_{i_1}U_{i_2}U_{i_3}\big)\, U^{\ell}U^{i_2}U^{i_3} \\
&\quad + \sum Tr\big(U_{i_1}(-C_{\ell i_2}(u_k))\, U_{i_2}U_{i_3}\big)\, U^{i_1}U^{\ell}U^{i_3} \\
&\qquad + \sum Tr\big(U_{i_1}U_{i_2}(-C_{\ell i_3}(u_k))\, U_{i_3}\big)\, U^{i_1}U^{i_2}U^{\ell} \\
&= -\sum Tr\big([U_k, U_\ell]\, U_{i_2}U_{i_3}\big)\, U^{\ell}U^{i_2}U^{i_3} \\
&\quad - \sum Tr\big(U_{i_1}\,[U_k, U_\ell]\, U_{i_3}\big)\, U^{i_1}U^{\ell}U^{i_3} \\
&\qquad - \sum Tr\big(U_{i_1}U_{i_2}\,[U_k, U_\ell]\big)\, U^{i_1}U^{i_2}U^{\ell}
\end{aligned}
$$

$$= -\sum \mathrm{Tr}([U_k, U_{i_1}] U_{i_2} U_{i_3}) U^{i_1} U^{i_2} U^{i_3}$$

$$-\sum \mathrm{Tr}(U_{i_1} [U_k, U_{i_2}] U_{i_3}) U^{i_1} U^{i_2} U^{i_3}$$

$$-\sum \mathrm{Tr}(U_{i_1} U_{i_2} [U_k, U_{i_3}]) U^{i_1} U^{i_2} U^{i_3}$$

$$= -\sum \mathrm{Tr}([U_k, U_{i_1} U_{i_2} U_{i_3}]) U^{i_1} U^{i_2} U^{i_3}$$

$$= -\sum \{\mathrm{Tr}(U_k U_{i_1} U_{i_2} U_{i_3}) - \mathrm{Tr}(U_{i_1} U_{i_2} U_{i_3} U_k)\} U^{i_1} U^{i_2} U^{i_3}$$

$$= -\sum 0 \; U^{i_1} U^{i_2} U^{i_3} = 0_{n \times n} .$$

Hence $U_k C_3 = C_3 U_k$ for every k $(1 \leqq k \leqq r)$. Thus we obtain that $C_3 \in Z(U(\mathfrak{g}))$. Q.E.D.

For example, it follows from Chevalley-Racah result that $Z(U(sl(3;\mathbb{C}))) \cong \mathbb{C}[C_2, C_3]$. By elementary calculation, we will show that $C_3 = -\frac{50}{27} I_3$ as follows.

Since $(U_1, U_2, U_3, U_4, U_5, U_6, U_7, U_8) = (E_{12}, E_{23}, E_{13}, E_{21}, E_{32}, E_{31}, E_{11}-E_{22}, E_{22}-E_{33})$ and $(U^1, U^2, U^3, U^4, U^5, U^6, U^7, U^8) = (E_{21}, E_{32}, E_{31}, E_{21}, E_{23}, E_{13}, (2/3)E_{11}-(1/3)E_{22}-(1/3)E_{33}, (1/3)E_{11}+(1/3)E_{22}-(2/3)E_{33})$,

we have

$U_1U_2=E_{13}$, $U_1U_4 = E_{11}$, $U_1U_7 = -E_{12}$, $U_1U_8 = E_{12}$, $U_2U_5 = E_{22}$, $U_2U_6 = E_{21}$,

$U_2U_8 = -E_{23}$, $U_3U_5 = E_{12}$, $U_3U_6 = E_{11}$, $U_3U_8 = -E_{13}$, $U_4U_7 = E_{21}$, $U_4U_3 = E_{23}$,

$U_4U_1 = E_{22}$, $U_5U_2 = E_{33}$, $U_5U_4 = E_{31}$, $U_5U_7 = -E_{32}$, $U_5U_8 = E_{32}$, $U_6U_1 = E_{32}$,

$U_6U_3 = E_{33}$, $U_6U_7 = E_{31}$, $U_7U_1 = E_{12}$, $U_7U_2 = -E_{23}$, $U_7U_3 = E_{13}$, $U_7U_4 = -E_{21}$,

$U_7U_8 = -E_{22}$, $U_8U_2 = E_{23}$, $U_8U_4 = E_{21}$, $U_8U_5 = -E_{32}$, $U_8U_6 = -E_{31}$, $U_8U_7 = -E_{22}$.

Now let us consider all the three-term products $U_{i_1}U_{i_2}U_{i_3} \neq O_3$ which have non-zero trace as follows.

$U_1U_2U_6 = E_{11}$, $U_1U_4U_7 = E_{11}$, $U_1U_7U_4 = -E_{11}$, $U_1U_8U_4 = E_{11}$, $U_2U_5U_7 = -E_{22}$,

$U_2U_5U_8 = E_{22}$, $U_2U_6U_1 = E_{22}$, $U_2U_8U_5 = -E_{22}$, $U_3U_5U_4 = E_{11}$, $U_3U_6U_7 = E_{11}$,

$U_3U_8U_6 = -E_{11}$, $U_4U_7U_1 = E_{22}$, $U_4U_3U_5 = E_{22}$, $U_4U_1U_7 = -E_{22}$, $U_4U_1U_8 = E_{22}$,

$U_5U_2U_8 = -E_{33}$, $U_5U_4U_3 = E_{33}$, $U_5U_7U_2 = -E_{33}$, $U_5U_8U_2 = E_{33}$, $U_6U_1U_2 = E_{33}$,

$U_6U_3U_8 = -E_{33}$, $U_6U_7U_3 = E_{33}$, $U_7U_1U_4 = E_{11}$, $U_7U_2U_5 = -E_{22}$, $U_7U_3U_6 = E_{11}$,

$U_7U_4U_1 = -E_{22}$, $U_7U_8U_7 = E_{22}$, $U_7U_8U_8 = -E_{22}$, $U_8U_2U_5 = E_{22}$, $U_8U_4U_1 = E_{22}$,

$U_8U_5U_2 = -E_{33}$, $U_8U_6U_3 = -E_{33}$, $U_8U_7U_7 = E_{22}$, $U_8U_7U_8 = -E_{22}$.

Hence C_3 is equal to $U^1U^2U^6 + U^1U^4U^7 - U^1U^7U^4 + U^1U^8U^4 - U^2U^5U^7 + U^2U^5U^8$

$+ U^2U^6U^1 - U^2U^8U^5 + U^3U^5U^4 + U^3U^6U^7 - U^3U^8U^6 + U^4U^7U^1 + U^4U^3U^5 - U^4U^1U^7 + U^4U^1U^8$

$- U^5U^2U^8 + U^5U^4U^3 - U^5U^7U^2 + U^5U^8U^2 + U^6U^1U^2 - U^6U^3U^8 + U^6U^7U^3 + U^7U^1U^4 - U^7U^2U^5$

$+ U^7U^3U^6 - U^7U^4U^1 + U^7U^8U^7 - U^7U^8U^8 + U^8U^2U^5 + U^8U^4U^1 - U^8U^5U^2 - U^8U^6U^3 + U^8U^7U^7$

$- U^8U^7U^8 = 0 - 1/3\, E_{22} - 2/3\, E_{22} + 1/3\, E_{22} + 1/3\, E_{33} - 2/3\, E_{33} + 0 - 1/3\, E_{33} + 0 - 1/3\, E_{33}$

$- 1/3\, E_{33} - 1/3\, E_{11} + 0 - 2/3\, E_{11} + 1/3\, E_{11} - 1/3\, E_{22} + 0 + 1/3\, E_{22} - 2/3\, E_{22} + 0 - 1/3\, E_{11}$

$- 1/3\, E_{11} - 1/3\, E_{22} + 1/3\, E_{33} - 1/3\, E_{33} - 2/3\, E_{11} + (4/27\, E_{11} + 1/27\, E_{22} - 2/27\, E_{33})$

$-(2/27\, E_{11} - 1/27\, E_{22} - 4/27\, E_{33}) - 2/3\, E_{33} + 1/3\, E_{11} - 1/3\, E_{22} - 1/3\, E_{11}$

$+ (4/27\, E_{11} + 1/27\, E_{22} - 2/27\, E_{33}) - (2/27\, E_{11} - 1/27\, E_{22} - 4/27\, E_{33})$

$= (- 2 + 4/27)\, I_3$.

Thus we obtain that $C_3 = - 50/27\ I_3$. Here we want to know what does it mean the number - 50/27 . The most likely result will be obtained by generalizing Weyl's dimension formula and Freudenthal formula.

REFERENCES

[1] Casimir H.B.G. ; Uber die Konstruction einer zu den irreduzibelen Darstellungen halbeinfacher kontinuierlicher Gruppen gehörigen Differentialgleichung, Proc. Kon. Acad. Amsterdam 34 (1931), S.144.

[2] Casimir H.B.G. and van der Waerden B.L. ; Algebraischer Beweis der vollständigen Reduzibilität der Darstellungen halbeinfacher Liescher Gruppen, Math. Ann. 111 (1935), 1-12.

[3] Gruber B. and O'Raifeartaigh L. ; S theorem and construction of the invariants of the semisimple compact Lie algebra, J. of Math. Physics 5 (1964), 1796-1804.

[4] Humphreys J.H. ; Introduction to Lie algebras and representation theory, G.T.M. 9 (Springer), 1972.

[5] Racah Giulio ; Group theory and spectroscopy, CERN Library 61-8 (1961), 44-56.

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