Modified Bernoulli nember $b_4 = -1/5760$ which appears as a coefficient both in Kontsevich integral of the unknot, and in Hirzebruch-Kodaira $\hat{A}$ genus of a complex manifold

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§1. Private historical introduction

Let $\rho = 1/2 \sum \alpha$ be half sum of positive roots of a semisimple complex Lie algebra G, and let (ρ, ρ) be the inner product that comes from Killing form. In 1963, Freudenthal de Vries proved a strange formula as follows.

$$24\ (\rho, \rho) = \dim G$$

In [4], the following fourth-order generalization was obtained ;

$$1/4!\ \sum (dim E^{\mu})(\mu,\rho)^4$$

$$= 1/4608\ (\dim E)\ \{(\lambda+\rho,\lambda+\rho)^2 + (\rho,\rho)^2\}$$

$$-1/2880\ (\dim E)\ \{\sum (\alpha,\lambda)^4 + 4 \sum(\alpha,\lambda)^3 (\alpha, \rho)$$

$$+ 6 \sum(\alpha,\lambda)^2 (\alpha,\rho)^2 + 4 \sum(\alpha,\lambda)\ (\alpha,\rho)^3\ \}$$

The author had no idea to explain why the rational coefficients 1/4608 and -1/2880 appear in the above formula in those days.
In 2014, we came across modified Bernoulli numbers via knot theory and via algebraic geometry, respectively.
In this report, I would like to introduce these two examples in the next section three.

§2. Historical introduction

In a posthumous work of Jakob Bernoulli (1654-1705), which was published in 1713, he calculated sums of k-th powers of natural numbers

$S^k(n) = 1^k + 2^k + ... + (n-1)^k$, where the Bernoulli numbers B_j have first appeared.

In fact, he obtained that

$S^k(n) = (k+1)^{-1} \sum(0 \leq j \leq k+1) Comb(k+1, j) B_j n^{k+1-j}$, where B_j are defined by

the following generating function ;

$x\ (e^x - 1)^{-1} = 1 - 1/2\ x + \sum(2 \leq j) B_j (j!)^{-1} x^j.$

Since $x\ (e^x - 1)^{-1} - 1 + (1/2)\ x$ is an even function, we have $B_{2m+1} = 0$ (m = 1, 2, 3, . . .). Moreover, the following table is very famous in number theorists.

$B_2 = 1/6, B_4 = -(30)^{-1}, B_6 = (42)^{-1}, B_8 = -(30)^{-1}, B_{10} = 5(66)^{-1}, B_{12} =$

$-691(2730)^{-1}$,

$B_{14} = 7/6, B_{16} = -3617(510)^{-1}, B_{18} = 43867(798)^{-1}, B_{20} = -174611(330)^{-1}.$

On the other, the generating function (which is seemed to be an origin of the Duflo-Killirov map) of the modified Bernoulli numbers is as follows.

$$1/2 \log ((e^{x/2} - e^{-x/2})x^{-1}) = \Sigma(0 \leq m) b_{2m} x^{2m}.$$

Differentiating the both sides of the above equation, we have

$$\Sigma(1 \leq m) b_{2m} 4m x^{2m} = x/2 \coth(x/2) - 1 = -1 + x/2 + x (e^{x} - 1)^{-1} =$$

$$\Sigma(1 \leq m) B_{2m}((2m)!)^{-1} x^{2m}.$$

Thus we know that

$$b_{2m} = (4m)^{-1}((2m)!)^{-1} B_{2m} (m = 1, 2, 3, ...).$$

In fact, $b_2 = (48)^{-1}, b_4 = -(5760)^{-1}, b_6 = (362880)^{-1}, ...$

§3. Main examples

In this section, we will raise two examples of occurrences of the fourth modified Bernoulli number $b_4 = -1/5760$.

(3.1) In the first edition (1956) of Hirzebruch's book [H;1] " Neue topologische Methoden in der algebraischen Geometrie " , it was used a terminology " A-genus "
In 1966, its English translation
[H;2] " Topological methods in algebraic geometry " , a modified terminology " Â-genus " was added in appendix. The reason why the changing words from " A-genus " to " Â-genus " is that new definition in a joint paper
[3] of Friedrich Hirzebruch and Kunihiko Kodaira, which had been published in 1957.

Here we will quote from p.204 in [3].

" Let $\hat{A}_0, \hat{A}_1, \hat{A}_2, \ldots$ be the multiplicative sequence of polynomials in the p_i belonging to the power series

$$1/2z^{1/2}\{ \mathrm{sh}\ (1/2\ z^{1/2})\}^{-1}.$$

We have

$$\hat{A}_0 = 1, \hat{A}_1 = -p_1/24, \hat{A}_2 = 2^{-7}(45)^{-1}\{ -4\ p_2 + 7(p_1)^2\}, \ldots"$$

For reader's convenience, let us recall the definition of Â-genus of a smooth oriented compact manifold M.

A set of polynomials $\{ K_j\}(; K_0 = 1, K_j = K_j(X_1, ..., X_j) \in Q[X_1, ..., X_j](j = 1, 2, 3, ...))$ is called a multiplicative sequence if and only if
every equation of formal power series (of z)

$$1 + \Sigma(1 \leq j)p_j z^j = \{1 + \Sigma(1 \leq j)p_j' z^j\}\{1 + \Sigma(1 \leq j)p_j'' z^j\}$$

always implies that $\Sigma(0 \leq j)K_j(p_1, ..., p_j)z^j$

$$= \{ \Sigma(0 \leq j)K_j(p_1', ..., p_j')z^j\}\{\Sigma(0 \leq j)K_j(p_1'', ..., p_j'')z^j\}.$$

Then $q(z) = 1 + K_1(1)z + K_2(1,0)z^2 + K_3(1,0,0)z^3 + ...$ is called the characteristic power series of a multiplicative sequence $\{ K_j\}$.

Conversely, it is known that
for every invertible formal power series $q \in 1 + zQ[[z]]$, there exists a multiplicative sequence $\{ K_j\}$ whose characteristic power series is equal to q , and that
the weighted degree of K_j is just equal to j , where the weighed degree of

$X_1^{e_1} X_2^{e_2} X_3^{e_3} ... X_j^{e_j}$ is defined by $e_1 + 2e_2 + 3e_3 + ... + je_j$.

In a case of $q = q(z) = 1/2\ z^{1/2}\{sinh(1/2z^{1/2})\}^{-1}$, and let

$p_j = p_j(M) \in H^{4j}(M, Z)$ be the Pontrjagin class of M .

Then $\hat{A}(M) = 1 + \Sigma(1 \leq j)K_j(p_1, ..., p_j)$ is called the Â-genus of M, and one can see that

$$\hat{A}(M) = 1 - 1/24\ p_1 + 1/5760(7(p_1)^2 - 4p_2) + ...$$

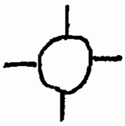
(3.2) According to the wheels theorem (11.23) in [2](p.330), it is known that the Kontsevich integral of the unknot is equal to

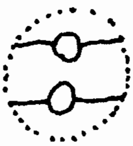
$$\exp\ (\ \Sigma(1 \leq m) b_{2m} W_{2m})$$

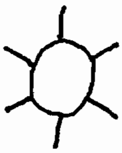
$$= 1 + 1/48\ W_2 - 1/5760\ W_4 + 1/4608\ W_2^2 + \ldots$$

, where W_{2m} denotes the wheel with 2m-spokes. In fact, the 2m-spokes wheel W_{2m} is the degree 2m uni-trivalent diagram of a 2m-gon with 2m legs as follows.

$W_2 =$ 

$W_4 =$ 

$W_2^2 =$ 

$W_6 =$ 

etc.

After a spider was spinning a regular web, watch a neighborhood of the center of the web, one often sees almost similar form to W_{24}.

§4. Sketch of graphical geometry

In this section, we will introduce basic relations in graphical geometry (, see [2]).

Figure (4.1) I H X -relation

Figure (4.2) S T U -relation

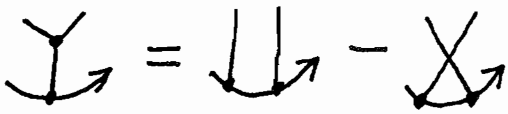


Figure (4.3) Jacobi relation [[X,Y], Z] = [X , [Y,Z]] - [Y , [X,Z]]

These three relations are equivalent to each other, because of smooth moving as follows.

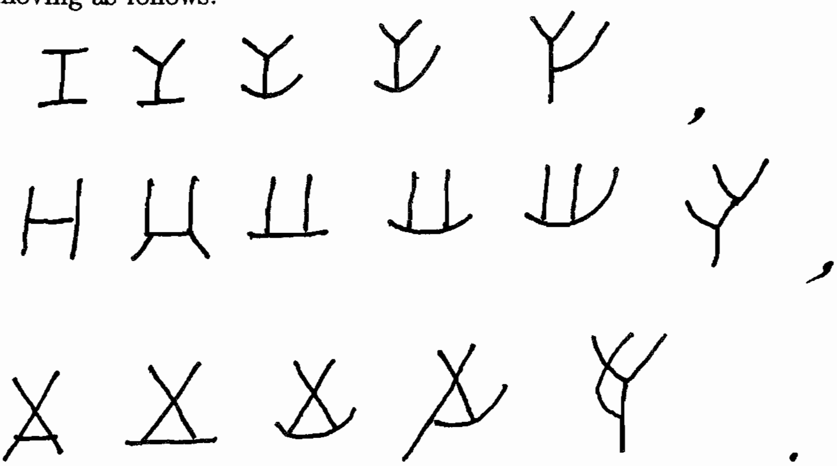
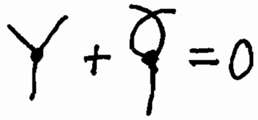
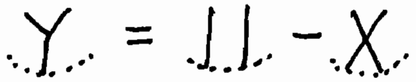


Figure (4.4) Antisymmetric relation ; [X,Y] + [Y,X] = 0



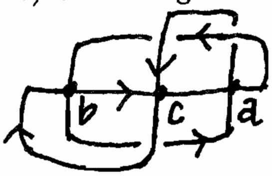
Since



and

, one sees that (4.2) implies (4.4).

Figure (4.5) Chord diagram of a singular knot



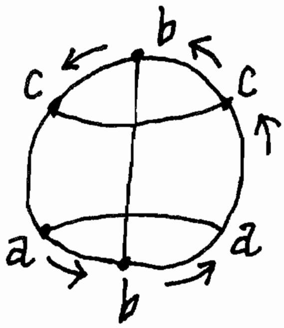


Figure (4.6) Skein relation of a knot invariant f

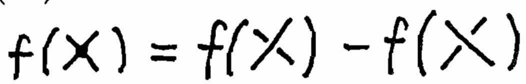


Figure (4.7) One-term relation ; [X,X] = 0

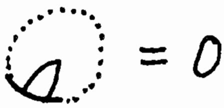
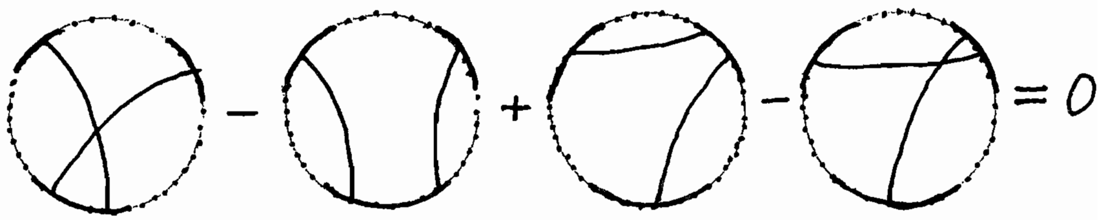
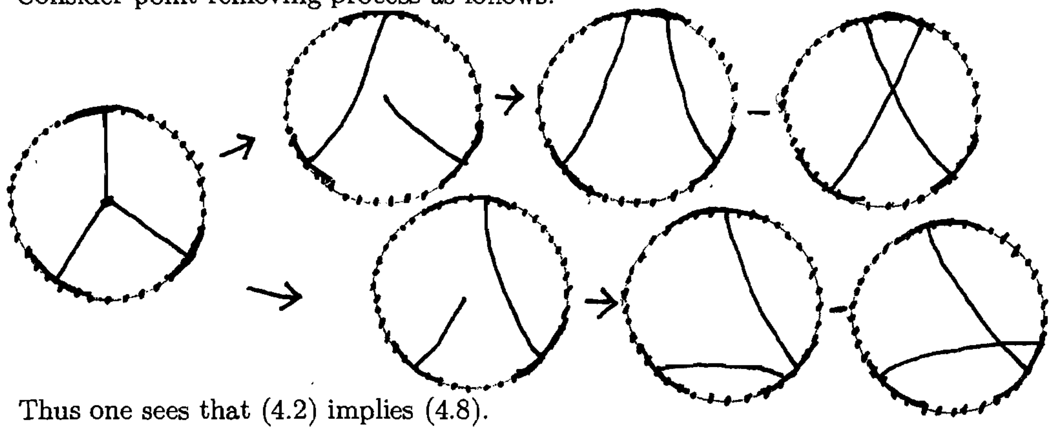


Figure (4.8) Four-term relation



Consider point removing process as follows.



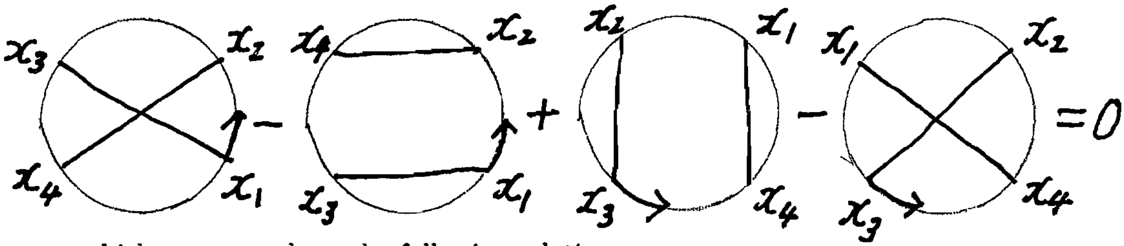
Thus one sees that (4.2) implies (4.8).

By the way, it is known (, see [1] (p.74, exercise 8)) that there exists a relation of Lie brackets as follows.

$$[[[X_1, X_2], X_3], X_4] + [[[X_2, X_1], X_4], X_3] + [[[X_3, X_4], X_1], X_2]$$

$$+ [[[X_4, X_3], X_2], X_1] = 0.$$

In graphical geometry, the following minimal four-term relation ;



, which corresponds to the following relation ;

$$[[[X_1, X_2], X_3], X_4] - [[[X_1, X_2], X_4], X_3] + [[[X_3, X_4], X_1], X_2]$$

$$- [[[X_3, X_4], X_2], X_1] = 0.$$

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