LEBESGUE 積分と DENJOY 積分の間に在る幾つかの積分について

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積分は、長さ・面積・体積の拡張概念であるとともに、微分の逆演算としての側面も持っている。前者の意味で積分を捉えるとき、現代では Lebesgue 積分を考えるのが一般的であり、収束定理や Fubini の定理等応用上十分な性質を持っている。しかし、Lebesgue 積分だけでは不十分な場合もあり、そのような場合には広義 Lebesgue 積分が用いられる。一方、後者の意味で積分を捉えるときは、Newton 積分を用いる。関数 f が Newton 積分可能であるとは、ある関数 F が存在して至る処で $F' = f$ が成り立つことである。至る処微分可能という条件はきつ過ぎるので、もう少し条件を緩め、可算個の点を除いて $F' = f$ が成り立つ、としたものを広義 Newton 積分可能という。

残念なことに、広義 Lebesgue 積分可能ではあるが Newton 積分可能でない関数や、広義 Newton 積分可能ではあるが Lebesgue 積分可能でない関数が存在する。微分方程式を素直に考えるのであれば、Newton 積分の方が適切であるが、Lebesgue 積分のような収束定理が存在しない。故に、この両者を包含し、尚且つ、収束定理を持つような積分が必要になる。この解が、Perron 積分であり、Denjoy 積分であり、Henstock-Kurzweil 積分である。これらは独立に定義が与えられているが、その後の研究で同値（積分可能関数の集合が一致する）であることが分かっている。また、応用上十分な収束定理も得られている。

Denjoy-Perron-Henstock-Kurzweil 積分によって、積分の持つ２つの側面を融合することが可能になった。だが、Lebesgue 積分と Newton 積分を包含するような最小の積分は存在しないのだろうか？　と言う疑問が出て来る。これに解を与えたのが、2000 年頃、B. Bongiorno によって C 積分と名付けられたものである。C 積分可能関数の集合は、Lebesgue 積分可能関数の集合及び Newton 積分可能関数の集合を包含し、逆に、C 積分可能関数は必ず Lebesgue 積分可能関数と Newton 積分可能関数との和で書ける。また、2006 年には、(B. Bongiorno の娘さんである) D. Bongiorno によって $\tilde{C}$ 積分と名付けられたものが与えられた。$\tilde{C}$ 積分可能関数の集合は、Lebesgue 積分可能関数の集合及び広義 Newton 積分可能関数の集合を包含し、逆に、$\tilde{C}$ 積分可能関数は必ず Lebesgue 積分可能関数と広義 Newton 積分可能関数との和で書ける。（彼女は言及していないが、$\tilde{C}$ 積分可能な関数の集合は広義 Lebesgue 積分可能な関数の集合も含む。）

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本稿では、以上の種々の積分について、その定義と包含関係について述べる。また、これらの積分の定義を俯瞰するとこれらの積分の間に別の積分の存在が見えてくる。これらの積分の存在について得られた最新の結果についても述べる。

1. Preliminaries

Throughout this paper we denote by $(\mathbf{L})(S)$, $(\mathbf{L}^*)(S)$ and $(\mathbf{D}^*)(S)$ the class of all Lebesgue integrable functions, the class of all improper Lebesgue integrable functions and the class of all restricted Denjoy integrable functions from a measurable set $S \subset \mathbb{R}$ into $\mathbb{R}$, respectively, and we denote by $|A|$ the measure of a measurable set A. We recall that a gauge δ is a function from an interval $[a, b]$ into $(0, \infty)$ and a δ-fine McShane partition of an interval $[a, b] \subset \mathbb{R}$ is a collection $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ of non-overlapping intervals $I_k \subset [a, b]$ and $x_k \in [a, b]$ satisfying $I_k \subset (x_k - \delta(x_k), x_k + \delta(x_k))$ and $\sum_{k=1}^{k_0} |I_k| = b - a$. If $\sum_{k=1}^{k_0} |I_k| \leq b - a$, then we say that the collection is a δ-fine partial McShane partition. Moreover, if $x_k \in I_k$ for any $k = 1, \ldots, k_0$, then a δ-fine McShane partition and a δ-fine partial McShane partition are called a δ-fine Perron partition and a δ-fine partial Perron partition, respectively. We say that a function f from an interval $[a, b]$ into $\mathbb{R}$ is Newton integrable if there exists a differentiable function F from $[a, b]$ into $\mathbb{R}$ such that $F' = f$ on $[a, b]$. We denote by $(\mathbf{N})([a, b])$ the class of all Newton integrable functions from $[a, b]$ into $\mathbb{R}$. In [3] B. Bongiorno, Di Piazza and Preiss gave a minimal constructive integration process of Riemann type, called the C-integral, which contains the Lebesgue integral and the Newton integral. Furthermore in [1–3] B. Bongiorno et al. gave some criteria for the C-integrability. We denote by $(\mathbf{C})([a, b])$ the class of all C-integrable functions from $[a, b]$ into $\mathbb{R}$. We say that a function f from an interval $[a, b]$ into $\mathbb{R}$ is improper Newton integrable if there exist a countable subset $N \subset [a, b]$ and a function F from $[a, b]$ into $\mathbb{R}$ such that $F' = f$ on $[a, b] \setminus N$. We denote by $(\mathbf{N}^*)([a, b])$ the class of all improper Newton integrable functions from $[a, b]$ into $\mathbb{R}$. In [4] D. Bongiorno gave a minimal constructive integration process of Riemann type, called the $\tilde{C}$-integral, which contains the Lebesgue integral and the improper Newton integral. Furthermore in [4] D. Bongiorno gave some criteria for the $\tilde{C}$-integrability. We denote by $(\tilde{C})([a, b])$ the class of all $\tilde{C}$-integrable functions from $[a, b]$ into $\mathbb{R}$. The improper Lebesgue integral, the C-integral and the $\tilde{C}$-integral are between the Lebesgue integral and the restricted Denjoy integral.

We know that the Lebesgue integral and the restricted Denjoy integral are equivalent to the McShane integral and the Henstock-Kurzweil integral, respectively. The McShane integral and the Henstock-Kurzweil integral are Riemann type integrals and these definitions are as follows.

Definition 1.1. A function f from an interval $[a, b]$ into $\mathbb{R}$ is McShane integrable if there exists a constant A such that for any positive number ε there exists a gauge δ

such that

$$\left| \sum_{k=1}^{k_0} f(x_k)|I_k| - A \right| < \varepsilon$$

for any δ-fine McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$. The constant A is the value of the McShane integral of f and we denote by

$$A = (MS) \int_{[a,b]} f(x)dx = (L) \int_{[a,b]} f(x)dx.$$

We denote by **(MS)**$([a, b])$ the class of all McShane integrable functions from $[a, b]$ into $\mathbb{R}$.

Definition 1.2. A function f from an interval $[a, b]$ into $\mathbb{R}$ is Henstock-Kurzweil integrable if there exists a constant A such that for any positive number ε there exists a gauge δ such that

$$\left| \sum_{k=1}^{k_0} f(x_k)|I_k| - A \right| < \varepsilon$$

for any δ-fine McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ with $x_k \in I_k$, that is, δ-fine Perron partition. The constant A is the value of the Henstock-Kurzweil integral of f and we denote by

$$A = (HK) \int_{[a,b]} f(x)dx = (D^*) \int_{[a,b]} f(x)dx.$$

We denote by **(HK)**$([a, b])$ the class of all Henstock-Kurzweil integrable functions from $[a, b]$ into $\mathbb{R}$.

In [5] D. Bongiorno showed a criterion for the improper Lebesgue integral as follows.

Theorem 1.1. *A function f from an interval $[a, b]$ into $\mathbb{R}$ is improper Lebesgue integrable if and only if there exist a constant A and a finite subset $N \subset [a, b]$ such that for any positive number ε there exists a gauge δ such that*

$$\left| \sum_{k=1}^{k_0} f(x_k)|I_k| - A \right| < \varepsilon$$

for any δ-fine McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying $x_k \in I_k$ whenever $x_k \in N$. Moreover

$$A = (L^*) \int_{[a,b]} f(x)dx.$$

The theorem above gives a Riemann type definition for the improper Lebesgue integral. In [1], see also [2, 3], B. Bongiorno gave the C-integral, which is also a Riemann type integral, as follows.

Definition 1.3. A function f from an interval $[a,b]$ into $\mathbb{R}$ is C-integrable if there exists a constant A such that for any positive number ε there exists a gauge δ such that

$$\left|\sum_{k=1}^{k_0} f(x_k)|I_k| - A\right| < \varepsilon$$

for any δ-fine McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$, where $d(I,x) = \inf_{y \in I} |y - x|$. The constant A is the value of the C-integral of f and we denote by

$$A = (C)\int_{[a,b]} f(x)dx.$$

We denote by $(\mathbf{C})([a,b])$ the class of all C-integrable functions from $[a,b]$ into $\mathbb{R}$.

In [4] D. Bongiorno gave the $\tilde{C}$-integral, which is also a Riemann type integral, as follows.

Definition 1.4. A function f from an interval $[a,b]$ into $\mathbb{R}$ is $\tilde{C}$-integrable if there exist a constant A and a countable subset $N \subset [a,b]$ such that for any positive number ε there exists a gauge δ such that

$$\left|\sum_{k=1}^{k_0} f(x_k)|I_k| - A\right| < \varepsilon$$

for any δ-fine McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying

(1) $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$;
(2) $x_k \in I_k$ whenever $x_k \in N$.

The constant A is the value of the $\tilde{C}$-integral of f and we denote by

$$A = (\tilde{C})\int_{[a,b]} f(x)dx.$$

We denote by $(\tilde{C})([a,b])$ the class of all $\tilde{C}$-integrable functions from $[a,b]$ into $\mathbb{R}$.

2. Definitions of new integrals

In this section firstly we define new integrals. By observing the definitions of the McShane, the improper Lebesgue in the sense of Theorem 1.1, the Henstock-Kurzweil integrals, C-integral and $\tilde{C}$-integral, we become aware of the following two integrals.

Definition 2.1. A function f from an interval $[a,b]$ into $\mathbb{R}$ is C*-integrable if there exist a constant A and a finite subset $N \subset [a,b]$ such that for any positive number ε there exists a gauge δ such that

$$\left|\sum_{k=1}^{k_0} f(x_k)|I_k| - A\right| < \varepsilon$$

for any δ-fine McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying

(1) $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$;

(2) $x_k \in I_k$ whenever $x_k \in N$.

The constant A is the value of the C*-integral of f and we denote by

$$A = (C^*) \int_{[a,b]} f(x)dx.$$

We denote by $(\mathbf{C}^*)([a, b])$ the class of all C*-integrable functions from $[a, b]$ into $\mathbb{R}$.

Definition 2.2. A function f from an interval $[a, b]$ into $\mathbb{R}$ is $\tilde{L}$-integrable if there exist a constant A and a countable subset $N \subset [a, b]$ such that for any positive number ε there exists a gauge δ such that

$$\left| \sum_{k=1}^{k_0} f(x_k)|I_k| - A \right| < \varepsilon$$

for any δ-fine McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying $x_k \in I_k$ whenever $x_k \in N$. The constant A is the value of the $\tilde{L}$-integral of f and we denote by

$$A = (\tilde{L}) \int_{[a,b]} f(x)dx.$$

We denote by $(\tilde{L})([a, b])$ the class of all $\tilde{L}$-integrable functions from $[a, b]$ into $\mathbb{R}$.

By the definitions of these integrals we obtain the following relations.

$$\begin{array}{ccccccc}
(\mathbf{N}) & \subset & & & (\mathbf{N}^*) & & (\mathbf{D}^*) \\
\cap & & & & \cap & & \\
(\mathbf{C}) & \subset & (\mathbf{C}^*) & \subset & (\tilde{C}) & & \| \\
\cup & & & & & \subset & \\
(\mathbf{MS}) & & \cup & & \cup & & (\mathbf{HK}) \\
\| & & & & & & \\
(\mathbf{L}) & \subset & (\mathbf{L}^*) & \subset & (\tilde{L}) & &
\end{array}$$

The above relations of inclusion are proper. We give some examples to check these. To show these, we provide the Saks-Henstock type lemmas. The following is the Saks-Henstock type lemma for the C*-integral.

Theorem 2.1. *If $f \in (\mathbf{C}^*)([a, b])$, then there exists a finite subset $N \subset [a, b]$ such that for any positive number ε there exists a gauge δ such that*

$$\sum_{k=1}^{k_0} \left| f(x_k)|I_k| - (C^*) \int_{I_k} f(x)dx \right| < \varepsilon$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying

(1) $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$;

(2) $x_k \in I_k$ *whenever* $x_k \in N$.

Proof. Since $f \in (\mathbf{C}^*)([a,b])$, there exists a finite subset $N \subset [a,b]$ such that for any positive number ε there exists a gauge δ such that

$$\left| \sum_{k=1}^{k_1} f(x_k)|I_k| - (C^*)\int_{[a,b]} f(x)dx \right| < \frac{\varepsilon}{4}$$

for any δ-fine McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_1\}$ satisfying

$$\sum_{k=1}^{k_1} d(I_k, x_k) < \frac{2}{\varepsilon}$$

and $x_k \in I_k$ whenever $x_k \in N$. Let $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ be a δ-fine partial McShane partition satisfying

$$\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$$

and $x_k \in I_k$ whenever $x_k \in N$, and let $\{I_k \mid k = k_0+1, \ldots, k_1\}$ be the sequence of intervals satisfying

$$\bigcup_{k=1}^{k_1} I_k = [a,b]$$

and $I_{k_2}^i \cap I_{k_3}^i = \emptyset$ if $k_2 \neq k_3$. Since f is C*-integrable on each I_k $(k = k_0+1, \ldots, k_1)$, there exists a gauge δ_k such that

$$\left| \sum_{\ell=1}^{\ell(k)} \left(f(x_{k,\ell})|I_{k,\ell}| - (C^*)\int_{I_{k,\ell}} f(x)dx \right) \right| < \frac{\varepsilon}{4(k_1 - k_0)}$$

for any δ_k-fine McShane partition $\{(I_{k,\ell}, x_{k,\ell}) \mid \ell = 1, \ldots, \ell(k)\}$ satisfying

$$\sum_{\ell=1}^{\ell(k)} d(I_{k,\ell}, x_{k,\ell}) < \frac{1}{\varepsilon(k_1 - k_0)}$$

and $x_{k,\ell} \in I_{k,\ell}$ whenever $x_{k,\ell} \in N$. Without loss of generality, it may be assumed that $\delta_k \leq \delta$ for any $k = k_0+1, \ldots, k_1$. Note that

$$\sum_{k=1}^{k_0} d(I_k, x_k) + \sum_{k=k_0+1}^{k_1} \sum_{\ell=1}^{\ell(k)} d(I_{k,\ell}, x_{k,\ell}) < \frac{1}{\varepsilon} + \sum_{k=k_0+1}^{k_1} \frac{1}{\varepsilon(k_1 - k_0)} = \frac{2}{\varepsilon}.$$

Therefore we obtain

$$\begin{aligned}
&\left|\sum_{k=1}^{k_0}\left(f(x_k)|I_k| - (C^*)\int_{I_k} f(x)dx\right)\right| \\
&\quad\le \left|\sum_{k=1}^{k_1} f(x_k)|I_k| - (C^*)\int_{[a,b]} f(x)dx\right| \\
&\qquad + \sum_{k=k_0+1}^{k_1}\left|\sum_{\ell=1}^{\ell(k)}\left(f(x_{k,\ell})|I_{k,\ell}| - (C^*)\int_{I_{k,\ell}} f(x)dx\right)\right| \\
&\quad< \frac{\varepsilon}{4} + \sum_{k=k_0+1}^{k_1}\frac{\varepsilon}{4(k_1-k_0)} = \frac{\varepsilon}{2}.
\end{aligned}$$

Moreover we obtain

$$\begin{aligned}
&\sum_{k=1}^{k_0}\left|f(x_k)|I_k| - (C^*)\int_{I_k} f(x)dx\right| \\
&\quad= \left|\sum_{f(x_k)|I_k|-(C^*)\int_{I_k} f(x)dx>0}\left(f(x_k)|I_k| - (C^*)\int_{I_k} f(x)dx\right)\right| \\
&\qquad + \left|\sum_{f(x_k)|I_k|-(C^*)\int_{I_k} f(x)dx<0}\left(f(x_k)|I_k| - (C^*)\int_{I_k} f(x)dx\right)\right| \\
&\quad< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

□

The following is the Saks-Henstock type lemma for the $\tilde{L}$-integral. The proof is similar to Theorem 2.1.

Theorem 2.2. *If $f \in (\tilde{L})([a,b])$, then there exists a countable subset $N \subset [a,b]$ such that for any positive number ε there exists a gauge δ such that*

$$\sum_{k=1}^{k_0}\left|f(x_k)|I_k| - (\tilde{L})\int_{I_k} f(x)dx\right| < \varepsilon$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying $x_k \in I_k$ whenever $x_k \in N$.

The Saks-Henstock type lemma for the improper Lebesgue integral also holds, see [5].

Theorem 2.3. *If $f \in (\mathbf{L}^*)([a,b])$, then there exists a finite subset $N \subset [a,b]$ such that for any positive number ε there exists a gauge δ such that*

$$\sum_{k=1}^{k_0} \left| f(x_k)|I_k| - (L^*)\int_{I_k} f(x)dx \right| < \varepsilon$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying $x_k \in I_k$ whenever $x_k \in N$.

We show that the above relations of inclusion are proper.

Theorem 2.4. *There exists a function f such that $f \in (\mathbf{C}^*)([0,1])$ but $f \notin (\mathbf{C})([0,1])$.*

Proof. Let f_1 be a function from $[0,1]$ into $\mathbb{R}$ defined by

$$f_1(x) = \begin{cases} (1-2x)\left(\sin\frac{1}{x(1-x)} - \frac{1}{x(1-x)}\cos\frac{1}{x(1-x)}\right), & \text{if } x \in (0,1), \\ 0, & \text{if } x \in \{0,1\}, \end{cases}$$

and let F_1 be a function defined by

$$F_1(x) = \begin{cases} x(1-x)\sin\frac{1}{x(1-x)}, & \text{if } x \in (0,1), \\ 0, & \text{if } x \in \{0,1\}. \end{cases}$$

Since f_1 is continuous on $(0,1)$ and

$$\lim_{\alpha\downarrow 0,\beta\uparrow 1}(L)\int_{[\alpha,\beta]} f_1(x)dx = \lim_{\alpha\downarrow 0,\beta\uparrow 1}(F_1(\beta) - F_1(\alpha)) = 0,$$

we obtain $f_1 \in (\mathbf{L}^*)([0,1])$ and hence $f_1 \in (\mathbf{C}^*)([0,1])$. However $f_1 \notin (\mathbf{C})([0,1])$. Indeed, assume that $f_1 \in (\mathbf{C})([0,1])$. Then by [2, Lemma 6] for any positive number ε with $\varepsilon < 1$ there exists a gauge δ such that

$$\sum_{k=1}^{k_0} |f_1(x_k)(b_k - a_k) - (F_1(b_k) - F_1(a_k))| < \varepsilon$$

for any δ-fine partial McShane partition $\{([a_k, b_k], x_k) \mid k = 1, \ldots, k_0\}$ satisfying

$$\sum_{k=1}^{k_0} d([a_k, b_k], x_k) < \frac{1}{\varepsilon}.$$

For any natural number n let

$$\begin{aligned} a_n &= \frac{1 - \sqrt{1 - \frac{4}{\frac{3}{2}\pi + 2n\pi}}}{2}, \\ b_n &= \frac{1 - \sqrt{1 - \frac{4}{\frac{\pi}{2} + 2n\pi}}}{2}. \end{aligned}$$

Note that $\{[a_n, b_n]\}$ is mutually disjoint and

$$\begin{aligned} F_1(a_n) &= -a_n(1-a_n) = -\frac{1}{\frac{3}{2}\pi + 2n\pi}, \\ F_1(b_n) &= b_n(1-b_n) = \frac{1}{\frac{\pi}{2} + 2n\pi}. \end{aligned}$$

Since the sequence $\{b_n(1-b_n) + a_n(1-a_n) \mid n \in \mathbb{N}\}$ is a strictly decreasing sequence tending to 0 and

$$0 < b_n(1-b_n) + a_n(1-a_n),$$
$$\sum_{n=1}^{\infty}(b_n(1-b_n) + a_n(1-a_n)) = \infty,$$

we can take a strictly increasing finite sequence $\{n(k) \mid k = 1, \ldots, k_0\}$ satisfying $b_{n(1)} < \delta(0)$ and

$$\varepsilon < \sum_{k=1}^{k_0}(b_{n(k)}(1-b_{n(k)}) + a_{n(k)}(1-a_{n(k)})) < \frac{1}{\varepsilon}.$$

Then $\{([a_{n(k)}, b_{n(k)}], 0) \mid k = 1, \ldots, k_0\}$ is a δ-fine partial McShane partition and satisfies

$$\sum_{k=1}^{k_0} d([a_{n(k)}, b_{n(k)}], 0) = \sum_{k=1}^{k_0} a_{n(k)} < \sum_{k=1}^{k_0}(b_{n(k)}(1-b_{n(k)}) + a_{n(k)}(1-a_{n(k)})) < \frac{1}{\varepsilon}.$$

However

$$\begin{aligned} &\sum_{k=1}^{k_0} |f_1(0)(b_{n(k)} - a_{n(k)}) - (F_1(b_{n(k)}) - F_1(a_{n(k)}))| \\ &= \sum_{k=1}^{k_0} |F_1(b_{n(k)}) - F_1(a_{n(k)})| \\ &= \sum_{k=1}^{k_0}(b_{n(k)}(1-b_{n(k)}) + a_{n(k)}(1-a_{n(k)})) \\ &> \varepsilon \end{aligned}$$

and hence it is a contradiction. □

Theorem 2.5. *There exists a function f such that $f \in (\tilde{C})([0,1])$ but $f \notin (\mathbf{C}^*)([0,1])$.*

Proof. Let f_2 be a function from $[0,1]$ into $\mathbb{R}$ defined by

$$f_2(x) = \begin{cases} n(n+1)f_1(n(n+1)x - n), & \text{if } x \in \left(\frac{1}{n+1}, \frac{1}{n}\right), n \in \mathbb{N}, \\ 0, & \text{if } x \in \left\{\frac{1}{n} \,\middle|\, n \in \mathbb{N}\right\} \cup \{0\}, \end{cases}$$

and let F_2 be a function defined by

$$F_2(x) = \begin{cases} F_1(n(n+1)x - n), & \text{if } x \in \left(\frac{1}{n+1}, \frac{1}{n}\right), n \in \mathbb{N}, \\ 0, & \text{if } x \in \left\{\frac{1}{n} \,\middle|\, n \in \mathbb{N}\right\} \cup \{0\}, \end{cases}$$

where f_1 and F_1 are the functions in Theorem 2.4. Since $F_2'(x) = f_2(x)$ for any $x \in \left(\frac{1}{n+1}, \frac{1}{n}\right)$, $n \in \mathbb{N}$, we obtain $f_2 \in (\mathbf{N}^*)([0,1])$ and hence $f_2 \in (\tilde{C})([0,1])$. However $f_2 \notin (\mathbf{C}^*)([0,1])$. Indeed, assume that $f_2 \in (\mathbf{C}^*)([0,1])$. Then by Theorem 2.1 there exists a finite subset $N \subset [0,1]$ such that for any positive number ε with $\varepsilon < 1$ there exists a gauge δ such that

$$\sum_{k=1}^{k_0} |f_2(x_k)(b_k - a_k) - (F_2(b_k) - F_2(a_k))| < \varepsilon$$

for any δ-fine partial McShane partition $\{([a_k, b_k], x_k) \mid k = 1, \ldots, k_0\}$ satisfying

(1) $\sum_{k=1}^{k_0} d([a_k, b_k], x_k) < \frac{1}{\varepsilon}$;

(2) $x_k \in [a_k, b_k]$ whenever $x_k \in N$.

Since N is finite, there exists a natural number p such that $\left[\frac{1}{p+1}, \frac{1}{p}\right] \cap N = \emptyset$. For any natural number n let

$$\begin{aligned} a_n &= \frac{1}{p+1} + \frac{1 - \sqrt{1 - \frac{4}{\frac{3}{2}\pi + 2n\pi}}}{2p(p+1)}, \\ b_n &= \frac{1}{p+1} + \frac{1 - \sqrt{1 - \frac{4}{\frac{\pi}{2} + 2n\pi}}}{2p(p+1)}. \end{aligned}$$

Note that $\{[a_n, b_n]\}$ is mutually disjoint and

$$\begin{aligned} F_2(a_n) &= -(p(p+1)a_n - p)(p + 1 - p(p+1)a_n) \\ &= -p(p+1)((p+1)a_n - 1)(1 - pa_n) \\ &= -\frac{1}{\frac{3}{2}\pi + 2n\pi}, \\ F_2(b_n) &= (p(p+1)b_n - p)(p + 1 - p(p+1)b_n) \\ &= p(p+1)((p+1)b_n - 1)(1 - pb_n) \\ &= \frac{1}{\frac{\pi}{2} + 2n\pi}. \end{aligned}$$

Since the sequence $\{p(p+1)(((p+1)b_n - 1)(1 - pb_n) + ((p+1)a_n - 1)(1 - pa_n)) \mid n \in \mathbb{N}\}$ is a strictly decreasing sequence tending to 0 and

$$0 < p(p+1)(((p+1)b_n - 1)(1 - pb_n) + ((p+1)a_n - 1)(1 - pa_n)),$$

$$\sum_{n=1}^{\infty} p(p+1)(((p+1)b_n - 1)(1 - pb_n) + ((p+1)a_n - 1)(1 - pa_n)) = \infty,$$

we can take a strictly increasing finite sequence $\{n(k) \mid k = 1, \dots, k_0\}$ satisfying $b_{n(1)} < \frac{1}{p+1} + \delta\left(\frac{1}{p+1}\right)$ and

$$\varepsilon < \sum_{k=1}^{k_0} p(p+1)(((p+1)b_{n(k)} - 1)(1 - pb_{n(k)}) + ((p+1)a_{n(k)} - 1)(1 - pa_{n(k)})) < \frac{1}{\varepsilon}.$$

Then $\left\{ \left([a_{n(k)}, b_{n(k)}], \frac{1}{p+1}\right) \,\middle|\, k = 1, \dots, k_0 \right\}$ is a δ-fine partial McShane partition and

$$\begin{aligned}
&\sum_{k=1}^{k_0} d\left([a_{n(k)}, b_{n(k)}], \frac{1}{p+1}\right) \\
&\quad = \sum_{k=1}^{k_0} \left(a_{n(k)} - \frac{1}{p+1}\right) \\
&\quad < \sum_{k=1}^{k_0} p(p+1)(((p+1)b_{n(k)} - 1)(1 - pb_{n(k)}) + ((p+1)a_{n(k)} - 1)(1 - pa_{n(k)})) \\
&\quad < \frac{1}{\varepsilon}.
\end{aligned}$$

However

$$\begin{aligned}
&\sum_{k=1}^{k_0} \left| f_2\left(\frac{1}{p+1}\right)(b_{n(k)} - a_{n(k)}) - (F_2(b_{n(k)}) - F_2(a_{n(k)})) \right| \\
&\quad = \sum_{k=1}^{k_0} |F_2(b_{n(k)}) - F_2(a_{n(k)})| \\
&\quad = \sum_{k=1}^{k_0} p(p+1)(((p+1)b_{n(k)} - 1)(1 - pb_{n(k)}) + ((p+1)a_{n(k)} - 1)(1 - pa_{n(k)})) \\
&\quad > \varepsilon
\end{aligned}$$

and hence it is a contradiction. □

Theorem 2.6. *There exists a function f such that $f \in (\tilde{L})([0,1])$ but $f \notin (\mathbf{L}^*)([0,1])$.*

Proof. Let f_3 be a function from $[0,1]$ into $\mathbb{R}$ defined by

$$f_3(x) = \begin{cases} f_1(n(n+1)x - n), & \text{if } x \in \left(\frac{1}{n+1}, \frac{1}{n}\right), n \in \mathbb{N}, \\ 0, & \text{if } x \in \left\{\frac{1}{n} \,\middle|\, n \in \mathbb{N}\right\} \cup \{0\}, \end{cases}$$

and let F_3 be a function defined by

$$F_3(x) = \begin{cases} \frac{1}{n(n+1)} F_1(n(n+1)x - n), & \text{if } x \in \left(\frac{1}{n+1}, \frac{1}{n}\right), n \in \mathbb{N}, \\ 0, & \text{if } x \in \left\{\frac{1}{n} \,\middle|\, n \in \mathbb{N}\right\} \cup \{0\}, \end{cases}$$

where f_1 and F_1 are the functions in Theorem 2.4. Then $f_3 \in (\tilde{L})([0,1])$ but $f_3 \notin$ $(\mathbf{L}^*)([0,1])$. Indeed, since f_3 is improper Lebesgue integrable on each $\left[\frac{1}{n+1}, \frac{1}{n}\right]$ and

$$(L^*) \int_{\left[\frac{1}{n+1}, \frac{1}{n}\right]} f_3(x)dx = 0,$$

by Theorem 1.1 there exists a finite subset $N_n \subset \left[\frac{1}{n+1}, \frac{1}{n}\right]$ such that for any positive number ε there exists a gauge δ_n such that

$$\left| \sum_{k=1}^{k_n} f_3(x_{n,k})|I_{n,k}| \right| < \frac{\varepsilon}{2^{n+1}}$$

for any δ_n-fine McShane partition $\{(I_{n,k}, x_{n,k}) \mid k = 1, \ldots, k_n\}$ of $\left[\frac{1}{n+1}, \frac{1}{n}\right]$ satisfying $x_{n,k} \in I_{n,k}$ whenever $x_{n,k} \in N_n$. It is obvious that $N_n = \left\{\frac{1}{n+1}, \frac{1}{n}\right\}$. Let

$$M_n = \max\left\{ |F_3(x)| \;\middle|\; x \in \left[\frac{1}{n+1}, \frac{1}{n}\right] \right\}.$$

It holds that $M_n = \frac{2}{n(n+1)} M_1$. Without loss of generality, it may be assumed that $(x - \delta_n(x), x + \delta_n(x)) \subset \left(\frac{1}{n+1}, \frac{1}{n}\right)$ for any $x \in \left(\frac{1}{n+1}, \frac{1}{n}\right)$. Let $N = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\} \cup \{0\}$, $\delta(x) = \delta_n(x)$ for any $x \in \left(\frac{1}{n+1}, \frac{1}{n}\right)$, $\delta\left(\frac{1}{n}\right) = \min\left\{\delta_n\left(\frac{1}{n}\right), \delta_{n-1}\left(\frac{1}{n}\right)\right\}$ for any $n \in \mathbb{N}$ with $n \geq 2$ and $\delta(0) < \frac{1}{p}$ with $M_p < \frac{\varepsilon}{2}$. Let $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ be a δ-fine McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying $x_k \in I_k$ whenever $x_k \in N$. Let $q = \min\left\{n \mid I_1 \cap \left[\frac{1}{n+1}, \frac{1}{n}\right) \neq \emptyset\right\}$. Then

$$\begin{aligned}
&\left| \sum_{k=1}^{k_0} f_3(x_k)|I_k| \right| \\
&= \left| f_3(0)|I_1| + \sum_{n=1}^{q} \sum_{I_k \subset \left[\frac{1}{n+1}, \frac{1}{n}\right]} f_3(x_k)|I_k| + \sum_{n=2}^{q} \sum_{\frac{1}{n} \in I_k} f_3\left(\frac{1}{n}\right)|I_k| \right| \\
&\leq |f_3(0)|I_1|| \\
&\quad + \left| \sum_{I_k \subset \left[\frac{1}{q+1}, \frac{1}{q}\right]} f_3(x_k)|I_k| + \sum_{\frac{1}{q} \in I_k} f_3\left(\frac{1}{q}\right) \left| I_k \cap \left[\frac{1}{q+1}, \frac{1}{q}\right] \right| \right| \\
&\quad + \sum_{n=2}^{q-1} \left| \sum_{\frac{1}{n+1} \in I_k} f_3\left(\frac{1}{n+1}\right) \left| I_k \cap \left[\frac{1}{n+1}, \frac{1}{n}\right] \right| \right. \\
&\quad + \sum_{I_k \subset \left[\frac{1}{n+1}, \frac{1}{n}\right]} f_3(x_k)|I_k|
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\frac{1}{n}\in I_k} f_3\left(\frac{1}{n}\right)\left|I_k\cap\left[\frac{1}{n+1},\frac{1}{n}\right]\right|\Bigg|\\
&+\left|\sum_{\frac{1}{2}\in I_k} f_3\left(\frac{1}{2}\right)\left|I_k\cap\left[\frac{1}{2},1\right]\right|+\sum_{I_k\subset\left[\frac{1}{2},1\right]} f_3(x_k)|I_k|\right|\\
\le\ & 0+\left|\sum_{I_k\subset\left[\frac{1}{q+1},\frac{1}{q}\right]} f_3(x_k)|I_k|+\sum_{\frac{1}{q}\in I_k} f_3\left(\frac{1}{q}\right)\left|I_k\cap\left[\frac{1}{q+1},\frac{1}{q}\right]\right|\right|+\sum_{n=2}^{q-1}\frac{\varepsilon}{2^{n+1}}+\frac{\varepsilon}{2^2}.
\end{aligned}
$$

By Theorem 2.3 we obtain

$$
\begin{aligned}
&\left|\sum_{I_k\subset\left[\frac{1}{q+1},\frac{1}{q}\right]} f_3(x_k)|I_k|+\sum_{\frac{1}{q}\in I_k} f_3\left(\frac{1}{q}\right)\left|I_k\cap\left[\frac{1}{q+1},\frac{1}{q}\right]\right|\right|\\
&\le \sum_{I_k\subset\left[\frac{1}{q+1},\frac{1}{q}\right]}\left|f_3(x_k)|I_k|-(L^*)\int_{I_k} f_3(x)dx\right|\\
&\quad+\sum_{\frac{1}{q}\in I_k}\left|f_3(x_k)\left|I_k\cap\left[\frac{1}{q+1},\frac{1}{q}\right]\right|-(L^*)\int_{I_k\cap\left[\frac{1}{q+1},\frac{1}{q}\right]} f_3(x)dx\right|\\
&\quad+\left|(L^*)\int_{\left(\bigcup_{I_k\subset\left[\frac{1}{q+1},\frac{1}{q}\right]} I_k\right)\cup\left(\bigcup_{\frac{1}{q}\in I_k} I_k\cap\left[\frac{1}{q+1},\frac{1}{q}\right]\right)} f_3(x)dx\right|\\
&<\frac{\varepsilon}{2^{q+1}}+M_q.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\left|\sum_{k=1}^{k_0} f_3(x_k)|I_k|\right| &< \frac{\varepsilon}{2^{q+1}}+M_q+\sum_{n=2}^{q-1}\frac{\varepsilon}{2^{n+1}}+\frac{\varepsilon}{2^2}\\
&< M_p+\sum_{n=1}^{\infty}\frac{\varepsilon}{2^{n+1}}\\
&< \varepsilon
\end{aligned}
$$

and hence $f_3\in(\tilde{L})([0,1])$. However, since it can be shown similarly to Theorem 2.5 that $f_3\notin(\mathbf{C}^*)([0,1])$, we obtain $f_3\notin(\mathbf{L}^*)([0,1])$. □

Theorem 2.7. *There exists a function f such that $f\in(\mathbf{C}^*)([0,1])$ but $f\notin(\mathbf{L}^*)([0,1])$.*

Proof. Let C be the Cantor set in $[0,1]$, let $\{(\alpha_p,\beta_p) \mid p \in \mathbb{N}\}$ be the sequence of all connected components of $[0,1]\setminus C$, let f_4 be a function from $[0,1]$ into $\mathbb{R}$ defined by

$$f_4(x) = \begin{cases} \frac{2(\alpha_p+\beta_p-2x)}{(\beta_p-\alpha_p)^2}\left(\frac{(x-\alpha_p)(\beta_p-x)}{(\beta_p-\alpha_p)^2}\sin\frac{(\beta_p-\alpha_p)^2}{(x-\alpha_p)(\beta_p-x)} - \cos\frac{(\beta_p-\alpha_p)^2}{(x-\alpha_p)(\beta_p-x)}\right), & \text{if } x \in (\alpha_p,\beta_p), p \in \mathbb{N}, \\ 0, & \text{if } x \in C, \end{cases}$$

and let F_4 be a function defined by

$$F_4(x) = \begin{cases} \frac{(x-\alpha_p)^2(\beta_p-x)^2}{(\beta_p-\alpha_p)^4}\sin\frac{(\beta_p-\alpha_p)^2}{(x-\alpha_p)(\beta_p-x)}, & \text{if } x \in (\alpha_p,\beta_p), p \in \mathbb{N}, \\ 0, & \text{if } x \in C. \end{cases}$$

Since $F_4'(x) = f_4(x)$ for any $x \in [0,1]$, we obtain $f_4 \in (\mathbf{N})([0,1])$ and hence $f_4 \in (\mathbf{C}^*)([0,1])$. However $f_4 \notin (\tilde{L})([0,1])$ and hence $f_4 \notin (\mathbf{L}^*)([0,1])$. We show $f_4 \notin (\tilde{L})([0,1])$. Assume that $f_4 \in (\tilde{L})([0,1])$. Then by Theorem 2.2 there exists a countable subset $N \subset [0,1]$ such that for any positive number ε there exists a gauge δ such that

$$\sum_{k=1}^{k_0} |f_4(x_k)(b_k-a_k) - (F_4(b_k)-F_4(a_k))| < \varepsilon$$

for any δ-fine partial McShane partition $\{([a_k,b_k],x_k) \mid k = 1,\ldots,k_0\}$ satisfying $x_k \in [a_k,b_k]$ whenever $x_k \in N$. Since N is countable and C is perfect, there exist $z \in C$ and $\{(\alpha_{p(q)},\beta_{p(q)}) \mid q \in \mathbb{N}\} \subset \{(\alpha_p,\beta_p) \mid p \in \mathbb{N}\}$ such that $z \notin N$ and $\left(\alpha_{p(q)}, \frac{\alpha_{p(q)}+\beta_{p(q)}}{2}\right) \subset [z, z+\delta(z))$ for any q. For any natural numbers q and n let

$$\begin{aligned} a_{q,n} &= \alpha_{p(q)} + \frac{(\beta_{p(q)}-\alpha_{p(q)})\left(1-\sqrt{1-\frac{4}{\frac{3}{2}\pi+2n\pi}}\right)}{2}, \\ b_{q,n} &= \alpha_{p(q)} + \frac{(\beta_{p(q)}-\alpha_{p(q)})\left(1-\sqrt{1-\frac{4}{\frac{\pi}{2}+2n\pi}}\right)}{2}. \end{aligned}$$

Note that $\{[a_{q,n}, b_{q,n}]\}$ is mutually disjoint and

$$\begin{aligned} F_4(a_{q,n}) &= -\frac{(a_{q,n}-\alpha_{p(q)})^2(\beta_{p(q)}-a_{q,n})^2}{(\beta_{p(q)}-\alpha_{p(q)})^4} \\ &= -\frac{1}{\left(\frac{3}{2}\pi+2n\pi\right)^2}, \\ F_4(b_{q,n}) &= \frac{(b_{q,n}-\alpha_{p(q)})^2(\beta_{p(q)}-b_{q,n})^2}{(\beta_{p(q)}-\alpha_{p(q)})^4} \\ &= \frac{1}{\left(\frac{\pi}{2}+2n\pi\right)^2}. \end{aligned}$$

Since $\{([a_{q,n}, b_{q,n}], z) \mid q, n \in \mathbb{N}\}$ is a δ-fine partial McShane partition and

$$\sum_{q=1}^{\infty}\sum_{n=1}^{\infty}|f_4(z)(b_{q,n}-a_{q,n})-(F_4(b_{q,n})-F_4(a_{q,n}))| = \sum_{q=1}^{\infty}\sum_{n=1}^{\infty}|F_4(b_{q,n})-F_4(a_{q,n})| = \infty,$$

there exists $\{([a_k, b_k], z) \mid k = 1, \dots, k_0\} \subset \{([a_{q,n}, b_{q,n}], z) \mid q, n \in \mathbb{N}\}$ such that

$$\sum_{k=1}^{k_0}|f_4(z)(b_k-a_k)-(F_4(b_k)-F_4(a_k))| > \varepsilon.$$

It is a contradiction. □

Theorem 2.8. *There exists a function f such that $f \in (\tilde{C})([0,1])$ but $f \notin (\tilde{L})([0,1])$.*

Proof. We show in the proof of Theorem 2.7 that $f_4 \in (\mathbf{N})([0,1])$ and hence $f_4 \in (\tilde{C})([0,1])$ but $f_4 \notin (\tilde{L})([0,1])$. □

Theorem 2.9. *There exists a function f such that $f \in (\mathbf{C}^*)([0,1])$ but $f \notin (\tilde{L})([0,1])$.*

Proof. We show in the proof of Theorem 2.7 that $f_4 \in (\mathbf{N})([0,1])$ and hence $f_4 \in (\mathbf{C}^*)([0,1])$ but $f_4 \notin (\tilde{L})([0,1])$. □

Theorem 2.10. *There exists a function f such that $f \in (\tilde{L})([0,1])$ but $f \notin (\mathbf{C}^*)([0,1])$.*

Proof. We show in the proof of Theorem 2.6 that $f_3 \in (\tilde{L})([0,1])$ but $f_3 \notin (\mathbf{C}^*)([0,1])$. □

3. Properties of the C*-integral

In this section we give a criterion for the C*-integrability.

Definition 3.1. Let F be an interval function on $[a,b]$ and let N be a finite subset of $[a,b]$. Then F is said to be C*-absolutely continuous on $E \subset [a,b]$ with respect to

N if for any positive number ε there exist a gauge δ and a positive number η such that

$$\sum_{k=1}^{k_0} |F(I_k)| < \varepsilon$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying

(1) $x_k \in E$ for any k;
(2) $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$;
(3) $x_k \in I_k$ whenever $x_k \in N$;
(4) $\sum_{k=1}^{k_0} |I_k| < \eta$.

We denote by $\mathbf{AC}_{C^*}(E, N)$ the class of all C*-absolutely continuous interval functions on E with respect to N. Moreover F is said to be C*-generalized absolutely continuous on $[a, b]$ if there exist a finite subset N and a sequence $\{E_m\}$ of measurable sets such that $\bigcup_{m=1}^{\infty} E_m = [a, b]$ and $F \in \mathbf{AC}_{C^*}(E_m, N)$ for any m. We denote by $\mathbf{ACG}_{C^*}([a, b])$ the class of all C*-generalized absolutely continuous interval functions on $[a, b]$.

Lemma 3.1. *If $F \in \mathbf{ACG}_{C^*}([a, b])$ and $E \subset [a, b]$ with $|E| = 0$, then there exists a finite subset $N \subset [a, b]$ such that for any positive number ε there exists a gauge δ such that*

$$\sum_{k=1}^{k_0} |F(I_k)| < \varepsilon$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying

(1) *$x_k \in E$ for any k;*
(2) *$\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$;*
(3) *$x_k \in I_k$ whenever $x_k \in N$.*

Proof. Since $F \in \mathbf{ACG}_{C^*}([a, b])$, there exist a finite subset $N \subset [a, b]$ and a sequence $\{E_m\}$ of measurable sets such that $\bigcup_{m=1}^{\infty} E_m = [a, b]$ and $F \in \mathbf{AC}_{C^*}(E_m, N)$ for any m. Therefore for any positive number ε and for any natural number m there exist a gauge δ_m and a positive number η_m such that

$$\sum_{k=1}^{k_0} |F(I_k)| < \frac{\varepsilon}{2^{m+1}}$$

for any δ_m-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying

(1) $x_k \in E_m$ for any k;
(2) $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$;
(3) $x_k \in I_k$ whenever $x_k \in N$;
(4) $\sum_{k=1}^{k_0} |I_k| < \eta_m$.

Since $|E \cap E_m| = 0$, there exists an open set $O_m \supset E \cap E_m$ such that $|O_m| < \eta_m$. Define $\delta_m^*(x) = \min\{\delta_m(x), d(O_m^c, x)\}$, where O_m^c is the complement of O_m. Then we obtain

$$\sum_{k=1}^{k_0} |F(I_k)| < \frac{\varepsilon}{2^{m+1}}$$

for any δ_m^*-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying (1), (2), (3) and (4). Define $\delta(x) = \delta_m^*(x)$ for any $x \in E \cap E_m$ $(m \in \mathbb{N})$. Then we obtain

$$\sum_{k=1}^{k_0} |F(I_k)| = \sum_{n=1}^{\infty} \sum_{x_k \in E_m} |F(I_k)| \leq \sum_{m=1}^{\infty} \frac{\varepsilon}{2^{m+1}} = \frac{\varepsilon}{2} < \varepsilon$$

for any δ-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying

(1) $x_k \in E$ for any k;
(2) $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$;
(3) $x_k \in I_k$ whenever $x_k \in N$.

□

Lemma 3.2. *If F is differentiable at $x \in [a, b]$, then for any positive number ε there exists a positive number δ such that*

$$|F(t) - F(s) - F'(x)(t - s)| < \varepsilon(2d([s, t], x) + t - s)$$

for any interval $[s, t] \subset (x - \delta, x + \delta) \cap [a, b]$.

Proof. Since F is differentiable at $x \in [a, b]$, there exists a positive number δ such that

$$|F(\xi) - F(x) - F'(x)(\xi - x)|| < \varepsilon|\xi - x|$$

for any $\xi \in (x - \delta, x + \delta) \cap [a, b]$. Therefore for any interval $[s, t] \subset (x - \delta, x + \delta) \cap [a, b]$ we obtain

$$\begin{aligned}
&|F(t) - F(s) - F'(x)(t - s)| \\
&\leq |F(t) - F(x) - F'(x)(t - x)| + |F(x) - F(s) - F'(x)(x - s)| \\
&< \varepsilon|t - x| + \varepsilon|s - x| \\
&= \varepsilon(2d([s, t], x) + t - s).
\end{aligned}$$

□

Theorem 3.1. *For any $F \in \mathbf{ACG}_{C^*}([a, b])$ there exists $\frac{d}{dx}F([a, x])$ for almost every $x \in [a, b]$, and there exists $f \in \mathbf{(C^*)}([a, b])$ such that $f(x) = \frac{d}{dx}F([a, x])$ for almost every $x \in [a, b]$ and*

$$F(I) = (C^*) \int_I f(x)dx$$

for any interval $I \subset [a, b]$.

Conversely the interval function F defined above for any $f \in (\mathbf{C}^*)([a,b])$ satisfies $F \in \mathbf{ACG}_{C^*}([a,b])$.

Proof. Note that, if $F \in \mathbf{ACG}_{\mathrm{C}^*}([a,b])$, then $F \in \mathbf{ACG}_\delta([a,b])$, see [7, Definition 9.14]. By [7, Theorem 9.17] there exists $\frac{d}{dx}F([a,x])$ for almost every $x \in [a,b]$. Let

$$E = \left\{ x \;\middle|\; \frac{d}{dx}F([a,x]) \text{ does not exist at } x \in [a,b] \right\}.$$

Then $|E| = 0$, and by Lemma 3.1 there exists a finite subset $N \subset [a,b]$ such that for any positive number ε with $\varepsilon < \frac{4}{b-a}$ there exists a gauge δ_1 such that

$$\sum_{k=1}^{k_0} |F(I_k)| < \frac{\varepsilon}{4}$$

for any δ_1-fine partial McShane partition $\{(I_k, x_k) \mid k = 1, \ldots, k_0\}$ satisfying

(1) $x_k \in E$ for any k;
(2) $\sum_{k=1}^{k_0} d(I_k, x_k) < \frac{1}{\varepsilon}$;
(3) $x_k \in I_k$ whenever $x_k \in N$.

If $x \notin E$, then by Lemma 3.2 there exists a positive number $\delta_2(x)$ such that

$$\left| F(t) - F(s) - \frac{d}{dx}F([a,x])(t-s) \right| < \frac{\varepsilon^2}{8}(2d([s,t],x) + t - s)$$

for any interval $[s,t] \subset (x - \delta_2(x), x + \delta_2(x)) \cap [a,b]$. Let

$$\delta(x) = \begin{cases} \delta_1(x), & \text{if } x \in E, \\ \delta_2(x), & \text{if } x \notin E, \end{cases}$$

and let

$$f(x) = \begin{cases} 0, & \text{if } x \in E, \\ \frac{d}{dx}F([a,x]), & \text{if } x \notin E. \end{cases}$$

Then we obtain

$$\begin{aligned}
\left|\sum_{k=1}^{k_0} f(x_k)|I_k| - F(I)\right| &\le \left|\sum_{x_k\in E} F(I_k)\right| + \left|\sum_{x_k\notin E} f(x_k)|I_k| - F(I_k)\right| \\
&\le \sum_{x_k\in E} |F(I_k)| + \sum_{x_k\notin E} |f(x_k)|I_k| - F(I_k)| \\
&< \frac{\varepsilon}{4} + \sum_{x_k\notin E} \frac{\varepsilon^2}{8}(2d(I_k,x_k) + |I_k|) \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon^2}{8}\cdot 2\cdot\frac{1}{\varepsilon} + \frac{\varepsilon^2}{8}(b-a) \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

for any interval $I \subset [a,b]$ and for any δ-fine McShane partition $\{(I_k,x_k) \mid k = 1,\ldots,k_0\}$ of I satisfying

(1) $\sum_{k=1}^{k_0} d(I_k,x_k) < \frac{1}{\varepsilon}$;
(2) $x_k \in I_k$ whenever $x_k \in N$.

Conversely let $f \in (\mathbf{C}^*)([a,b])$ and let

$$F(I) = (C^*)\int_I f(x)dx$$

for any interval $I \subset [a,b]$. For any natural number m let $E_m = \{x \mid x \in [a,b], |f(x)| \le m\}$. Then $\bigcup_{m=1}^{\infty} E_m = [a,b]$. We show that $F \in \mathbf{AC}_{C^*}(E_m, N)$, where N is an excepting finite subset of $[a,b]$ in the definition of the C*-integral of f. Let ε be a positive number. By Theorem 2.1 there exists a gauge δ such that

$$\sum_{k=1}^{k_0} |f(x_k)|I_k| - F(I_k)| < \frac{\varepsilon}{2}$$

for any δ-fine partial McShane partition $\{(I_k,x_k) \mid k = 1,\ldots,k_0\}$ satisfying

(1) $\sum_{k=1}^{k_0} d(I_k,x_k) < \frac{1}{\varepsilon}$;
(2) $x_k \in I_k$ whenever $x_k \in N$.

Let $\eta = \frac{\varepsilon}{2m}$. If $x_k \in E_m$ for any k and $\sum_{k=1}^{k_0} |I_k| < \eta$, then we obtain

$$\begin{aligned}
\sum_{k=1}^{k_0} |F(I_k)| &\le \sum_{k=1}^{k_0} |f(x_k)||I_k| + \sum_{k=1}^{k_0} |f(x_k)|I_k| - F(I_k)| \\
&< m\sum_{k=1}^{k_0} |I_k| + \frac{\varepsilon}{2} \\
&< \varepsilon.
\end{aligned}$$

□

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