POISSON' "*SLOUGHS*" IN HIS FINAL WORKS IN LIFE,

"*A STUDY OF MATHEMATICAL PHYSICS.*"

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1. Preface

1.1. Remark on continuum for heat theory and fluid dynamics.

[1] Duhamel [5] comments on the continuum and Poisson's paper [21] :

> We explain afterward how he do with Mr. Poisson obtain the same equation with Navier has made known in 1821, with talking the molecular actions, and in considering the corps as continue. This method inspecting the molecular actions is originally due to Laplace, who has deduced from this a nice theory of capillary action. Mr. Navier has obtained afterward the nice idea to deduce the theory of elastic solid ; however, both of the mathematicians have supposed the molecules of adjacent corps, and Poisson is the first of coincidence with calculations with the physical structures. In addition to, although the hypotheses of continuum theory have been actually so inexact, however, have played big roles in the science. In the roles, have played, the theories by Mr. Laplace have welcomed by the researchers. This observation on the molecular activities, in the bulk of special problems, above all, in theory of the elastic bodies, it has the very countless merits to have to *sweep out the all special hypotheses.* Mr. Poisson emphasizes the merit of this method ; we will reproduce textually this passage from his Mémoire. [5, pp.98-99] (trans. and italics mine.)

We would like to point out Poisson' sloughs in his final works in life :

1. He proposes the cause of rise/fall of capillary surface is due to the variation of density. Today's explanation is due to surface tension. (Part 1.)
2. Another equation of fluid dynamics, which is the original of the Navier Stokes equations. (Part 2.)
3. He conjectures the proof on the exact differential will be defect. (Part 2.)
4. The difference between Lagrange's series and the Fourier Series. (Part 2.)
5. The celestial mechanics in conformity to the mathematical physics. (Part 2.)
6. Another equation of heat different from Fourier. (Part 3.)

In the table 1, Poisson's second books [24, 25] and third book [26] seem to be contradict in the order of publishing year on describing title pages, (Poisson says the second book is [26]), however, he explains as follows :

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[1] Siméon Denis Poisson : Born 21/June/1781 at Pithiviers, dead 25/April/1840 at Sceaux, Poisson enters l'Ecole Polytechnique in 1798 and there, will make career as professor. His works are numerous (almost 400 published) and amount specially to applied mathematics and to the physics. (HP of fdp : Fédération Denis Poisson).

This *Theorie mathématiques de la chaleur*, (*Mathematical theory of heat*) will form the second part of *Un Traité de Physique mathématiques*, [2] (*A Study of Mathematical Physics*), where I propose to consider successively, without hesitation for any order preventing the progress, the diverse questions of the physics to which I will apply the analysis. The primary part of this *Traité* is the *Nouvelle théorie de l'Action capillaire*, the (*New Theorie of the Capillary Action*), published in 1831. [26, p.6]. [3]

Consequently, he doesn't hesitate for 'any order' preventing the progress, where, we think, he seems to intends to slough from the old-fashoned order in the wide meaning, because he struggles for the truth in rivalry relations in his life. So we use our title from this phrase. [4] [5]

TABLE 1. The three books consisted of *A Study of Mathematical Physics.*

		1	2	3
	name and bibliography	New theory of Capillary Action 1831 [23]	Study of Mechanics 1833 [24, 25]	Analytical Theory of Heat 1835 [26]
1	pages+figures	326+1	[24] : 696+4, [25] : 782+3 total : 1478+7	543+1
				2347+9
2	rivalry & preceding studies	Laplace 1805 [12], Gauss 1830 [8]	Langarnge 1788 [10], Laplace 1798-05 [11]	Fourier 1822 [6]
3	newness & uniqueness	• rising by density variation • adaptation to continuum	• mathematical principles of mechanics • analysis of exact differential • Lagrange's summation & Fourier's series • uptodate astronomy	based on this general hypothesis of a molecular radiation. (cf. § 7.)

[2](⇓) There doesn't exist any book entitled this name. He published also *Traité de Mécanique*, in 1811. cf. [19], however, this is neither identical with [19] in respect to the title, nor the publishing date.

[3](⇓) cf. [23].

[4]'slough' means the molting of a cicada which comes out his shell or a human's breaking with scientific conventions. Poisson says in [26, p.6] as follows : This Theorie mathmatiques de la chaleur, (Mathematical theory of heat) will form the second part of *Un Trait de Physique mathmatiques*, (A Study of Mathematical Physics), where I propose to consider successively, without hesitation for any order preventing the progress, the diverse questions of the physics to which I will apply the analysis. The primary part of this Trait is the Nouvelle thorie de l ' Action capillaire, the (New Theorie of the Capillary Action), published in 1831.

[5]To establish a time line of these contributors, we list for easy reference the year of their birth and death: Kepler(1571-1630), Newton(1642-1727), Daniel Bernoulli(1700-82), Euler(1707-83), d'Alembert(1717-83), Lagrange(1736-1813), Laplace(1749-1827), Legendre (1752-1833), Fourier(1768-1830), Gauss(1777-1855), Poisson(1781-1840), Bessel(1784-1846), Navier(1785-1836), Cauchy(1789-1857), Dirichlet(1805-59), Stokes(1819-1903), Riemann(1826-66).

2. Part 1. *New Theory of the Capillary Action.*

2.1. **The general conception of capillary action.** [6]

Poisson discuss the attractive and repulsive forces in the hydrostatics, in the hydrodynamics, and in the heat theory, citing his paper [21]. cf. [23, p.30]. Poisson mentions p as $p = \varpi + \Delta$, he defines $f(r)$ the measure of the molecular action in the distance r and related with the unit of the volume.

> Hence, to satisfy the conditions of preceding article, the sum which p represents can't be reduced to an integral, and it must be equal with two terms ϖ and Δ of its complete value. Although the smallness of ε, the latter term can effectively become comparable and same measure than the former, when the two attractive and repulsive forces, which dues to $f(r)$, are mutually, extremely great in comparison to the difference. We show this point the development and the examples which I have given in my memoir on the general equation of the equilibrium and the motion of the solid elastic corps and fluids (cf. [21]). [23, p.30] [7]

In the other hand, Poisson cites [21] in his book on the capillary action [23], discussing the same theme of the development and example of the two attractive and repulsive forces. Although deducing into the same results of the fundamental formula respectively as follows, Poisson asserts his own discussion on the attractive and repulsive forces, whose method comes from the essential conception among the hydrodynamics, hydrostatics and heat theory. Followings coincide respectively in expression of the formulae.

Laplace [12, p.19] (R and R' are the radii of the priciple curvatures, respectively) :

$$\frac{1}{R} + \frac{1}{R'} = \frac{(1+q^2)\frac{dp}{dx} - pq\left(\frac{dp}{dy} + \frac{dq}{dx}\right) + (1+p^2)\frac{dq}{dy}}{(1+p^2+q^2)^{\frac{3}{2}}}, \quad p = \frac{dz}{dx}, \quad q = \frac{dz}{dy} \tag{1}$$

Gauss [8, p.64-65] (R and R' are the same with Laplace) :

$$\begin{aligned} \frac{d\xi}{dx} + \frac{d\eta}{dy} &= -\zeta^3\Big[\frac{d^2z}{dx^2}\Big\{1 + \Big(\frac{dz}{dy}\Big)^2\Big\} - \frac{2d^2z}{dx.dy}.\frac{dz}{dx}.\frac{dz}{dy} + \frac{d^2z}{dy^2}\Big\{1 + \Big(\frac{dz}{dx}\Big)^2\Big\}\Big] = \frac{1}{R} + \frac{1}{R'}, \\ where, \quad \zeta^3 &= \Big[1 + \big(\frac{dz}{dx}\big)^2 + \big(\frac{dz}{dy}\big)^2\Big]^{-\frac{3}{2}}. \end{aligned} \tag{2}$$

Poisson [23, p.61, p.99] (λ and λ' are the radii of the priciple curvatures, respectively) :

$$\frac{1}{\lambda} + \frac{1}{\lambda'} = \frac{\left[1 + \left(\frac{dz}{dy}\right)^2\right]\frac{d^2z}{dx^2} - 2\frac{dz}{dx}\frac{dz}{dy}\frac{d^2z}{dxdy} + \left[1 + \left(\frac{dz}{dx}\right)^2\right]\frac{d^2z}{dy^2}}{\left[1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right]^{\frac{3}{2}}} \tag{3}$$

These fundamental formulae are conventionally deduced after discussing personally the molecular activity between the attractive and repulsive forces, even if it were differed with each other. Poisson's one is based on the essential concept dues to [21].

Their form is the same with that of the equations of the *Mécanique céleste* (Celestial mechanics) ; however, the expressions in definite integrals of two special constants which they include are very different, so that their numerical values would be equally if, instead of determining it with the experience, we would be capable to calculate them directly owing to their analytic expressions, this one, which would be necessary that we would

[6]($\Downarrow$) The original title is : *Nouvelle Théorie de l'Action Capillaire.* [23]

[7]For example, cf. [21, pp. 98-99, p.134, pp.170-1]

know the law of the action of the tube on the liquid and of liquid on itself.

With the rules known of the calculation of the variation, we determine the surface unknown of the liquid which makes this sum a *minimum*, and as we see, we find at once, the general equation of this surface and the equation particular to its contour, this one, which is the characteristic merit of the method which Mr. Gauss has followed. But, this great prodigious mathematician having started from the similarly given physics with Laplace, and having no more consider the variation of the density at the extremity of the liquid, which he has regarded, in contrary, as incompressible in all the parts, the objections which is structured against the theory of Laplace applies equally to his, which isn't different with the other from the manner to formulate the equations of the equilibrium.

General consequence which we will make from our theory, it is here that the phenomena of the capillarity are due to the molecular action, modified, not only with the curvature of the surface, as Laplace has discussed, but also with the particular state of the liquids at their extremities.

TABLE 2. The three papers/a book on the capillary action

		1	2	3
	Name and bibliography	Laplace(1749-827) 1798-05 [11] 1805 [12], et.al.	Gauss(1777-855) 1830 [8]	Poisson(1781-840) 1831 [23]
1	language, pages	French, 78	Latin, 49	French, 302
2	restrictions	incompressible fluid	according to Laplace' physics, incompressible fluid	
3	composition of capillary forces	• attraction • repulsion (after 1819)	• gravity • the attractive force • for these forces, we will designate the $\prec$ *characteristic* $F \succ$ such that the inverse-directional distance is used.	• universal attraction • molecular attraction • calorific repulsion (§ 129)
4	mathematical newness	• two special constants • equation of surface using principal radii of curvature • adaptation to continuum	• introduction of variation problem from Lagrange (§ 18) • analysis from geometry (§ 20) • comparison of efficiency of methods between analysis and geometry (§ 25) • reduction from sextuplex integral to quadruplex integral (§ 16) • principle of virtual velocity	• adaptation in both theory and practice to continuum • analysis of fluidity(§ 62) • the difference between fluid and solid corps.(§ 131) • principle of heat theory (§ 129) • calculation in aid of elliptic function by Legendre • point of arête vive (§ 112 and ff.) • point of inflection (§ 54 and ff.). • reduction from multiplex integrals such as quitiplex (§ 17), sextuplex (§ 18)

2.2. **The modeling and proof of rise/fall in liquid.**

We discuss the problem of capillary action by Poisson (1831) from the history of mathematical physics, or, the modeling and calculation of the rise/fall of the liquid in the neighborhood of wall.

He supposes the mutual action of attraction between the molecules, $\rho^2\varphi(r)\omega\omega' dsds'$ with the function $\varphi(r)$ of distance r between two molecules. He separates the domain of the liquid into four parts C, C', D and D', of which the two are near the wall : C locates over C', and other two, in the liquid, D locates over D'. He seeks the unknown Δ from the $2R' - R \equiv \Delta$, where, R and R' the actions from the liquid and from the wall. Under the condition of constant density of liquid,

$$R = \rho^2 \int\int\int\int\int\int \varphi(r)\frac{z+z'}{r}(1-ku)(1+k'u')dzdz'dudu'dsds',$$

in putting $r^2 = x^2 + (u+u')^2 + (z+z')^2$ and putting $k = k' = 1$,

$$q \equiv 2\rho^2 \int\int\int\int\int \varphi(r)\frac{z+z'}{r}dzdz'dudu'dx, \quad q' \equiv 2\rho\rho' \int\int\int\int\int \varphi'(r)\frac{z+z'}{r}dzdz'dudu'dx,$$

where, q and q' the quantities, ρ and ρ' densities of two material, dz, dz', du, du', dx, each elements of the distances. Using c the contour and $R = \int qds = cq$, and integrating the function, he calculates the quantities of action Q, Q', P defined Q in D, Q' in D' and P in C', under the condition of equilibrium $Q + Q' + P = 0$ in D, and, he gets $P = -2cq$, $Q' = R = cq$, $Q = \Delta = cq$. (On the Q, he shows another direct method.)

By his hypothesis, it turns finally $q = q'$, $\rho = \rho'$, because of the constant density, namely, it means that the materials are equal between the tube and liquid. From this contradiction, he concludes the rise/fall dues to the abrupt change of variation in density of liquid near the wall.

If we calculate Q without using the equilibrium in D, then we get as follows.

$$\begin{aligned} Q &= \rho^2 \int\int\int\int\int\int \varphi(r)\frac{z+z'}{r}dzdz'dudu'dxds, \\ r^2 &= x^2 + (u+u')^2 + (z-z')^2. \end{aligned}$$

in conserving all the notations of the number cited, and integrating in respect to $z$and z', from the plane GF to its tangential plane.

$$Z = \int_0^{y+\theta u}\int_0^{y-\theta u} \varphi(r)\frac{z'-z}{r}dzdz'.$$

from the above, we conclude

$$\int\int\int Zdudu'dx = 2\int_0^\infty \Phi(r')rdr \int_0^\infty\int_0^\infty \frac{dud\zeta}{(1+\zeta^2)\left[1+u^2(1+\zeta^2)\right]},$$

namely,

$$\int\int\int Zdudu'dx = \frac{\pi\theta}{\sqrt{1+\theta^2}}\int_0^\infty r\Phi'(r)dr,$$

in effectuating the integration relative to u, and next, that which responds to ζ.

Owing to this reduction of the integral in respect to z, z', u, u', x, and in putting for θ its value $-\cot\omega$, the expression of Q will turn into

$$Q = -\pi\rho^2\int_0^\infty r\Phi'(r)dr. \int \cos\omega ds. \tag{4}$$

In integrating by parts, it turns into

$$\int_0^\infty r\Phi'(r)dr = \frac{1}{2}\int_0^\infty r^3\Phi(r)dr = \frac{1}{8}\int_0^\infty r^4\varphi(r)dr\ ;$$

from the above, we conclude with

$$q = \frac{\pi\rho^2}{8}\int_0^\infty r^4\varphi(r)dr,$$

from (4), we get :

$$Q = -q\int \cos\omega ds. \qquad Q = -cq\cos\omega,$$

where, $\int ds = c$.

3. Part 2. *Study of Mechanics.*

The *material* is all this one, which can affect our sense of a certain manner. The *corps* are the portion of material limited in all sense, and which have, in consequence, a *form* and a *volume* determined. We call *mass* of a corps, the quantity of material of which it is composed.

A *material point* is a corps infinitely small in all the dimensions ; so that the length of all the line composed in it interior, is infinitely small, namely, less than all length which we can assign. We can regard a corps of finite dimensions, as a assemble of an infinity of material points and its mass as the sum of all their masses infinitely small.

3.1. **Another equation of fluid dynamics, which is the original of the Navier Stokes equations.**

$$(7\text{-}9)_{Pf} \quad \begin{cases} \rho(X - \frac{d^2x}{dt^2}) = \frac{d\varpi}{dx} + \beta(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}), \\ \rho(Y - \frac{d^2y}{dt^2}) = \frac{d\varpi}{dy} + \beta(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}), \\ \rho(Z - \frac{d^2z}{dt^2}) = \frac{d\varpi}{dz} + \beta(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}) \\ \text{ where } \varpi = p + \frac{\alpha}{3}(K+k)(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}), \end{cases} \tag{5}$$

$$\Rightarrow^* \quad \begin{cases} \rho(\frac{Du}{Dt} - X) + \frac{dp}{dx} + \alpha(K+k)\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) + \frac{1}{3}\alpha(K+k)\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \\ \rho(\frac{Dv}{Dt} - Y) + \frac{dp}{dy} + \alpha(K+k)\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right) + \frac{1}{3}\alpha(K+k)\frac{d}{dy}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \\ \rho(\frac{Dw}{Dt} - Z) + \frac{dp}{dz} + \alpha(K+k)\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\right) + \frac{1}{3}\alpha(K+k)\frac{d}{dz}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \end{cases}$$

($\Downarrow$) Here, $\alpha(K+k)$ is the constant to the tensor function with the main axis (the normal stress) of Laplacian. $\frac{1}{3}\alpha(K+k)$ corresponds to the coefficient of grad.div term. In today's NS equations, the ratio of coefficient attached to the term of the tensor function with the main axis (the normal stress) of Laplacian to that of grad div : $\frac{\text{coefficient of tensor}}{\text{coefficient of grad div}} = 3$, like Poisson deduced in $(7\text{-}9)_{Pf}$ and Stokes' $(12)_S$ through the tensor by Saint-Venant. By Prandtl [28, p.259] in 1934, we had have to wait by the time, when including this ratio of two coefficients, as what is called the NS equations were expressed in fluid equation. ($\Uparrow$) Stokes pointed out the coincidence with Poisson using the correspondence:

$\varpi = p + \frac{\alpha}{3}(K+k)\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)$ which then gives $\nabla\varpi = \nabla p + \frac{\beta}{3}\nabla(\nabla\cdot\mathbf{u})$. Stokes also commented:

> The same equations have also been obtained by Navier in the case of an incompressible fluid (Mém. de l'Académie, t. VI. p.389) [8], but his principles differ from mine still more than do Poisson's. [29, p.77, footnote]

He further stated:

> Observing that $\alpha(K+k) \equiv \beta$, this value of ϖ reduces Poisson's equation $(7\text{-}9)_{Pf}$ (=(5) in our renumbering) to the equation $(12)_S$ of this paper.

3.2. **Slough from Lagrange to Fourier.**

In 1807, Fourier proposes his first paper on the heat theory to French Academy, howeevr Lagrange (the then one of the commision, [9] or chief judges of this arena) rejects this paper, and Poisson obeys to his teacher (Lagrange), and objects that Fourier's deduction dues to Lagrange's originality. But Today, Poisson acknowledges Fourier's originality. We show Poisson's introduction in [24] the comparison and differences between Lagrange (§ **325**) and Fourier (§ **328**). cf. [14, p.47]

§ **325**. (Lagrange's formula.) We go now to demonstrate the formula of Lagrange, cited previously.

For this, let consider the quantity $\frac{1-h^2}{1-2h\cos\theta+h^2}$, which is a rational fraction with respect to h, and in which θ designate a real angle. Its development following the powers of h will be $1+2h\cos\theta+2h^2\cos 2\theta+2h^3\cos 3\theta+2h^4\cos 4\theta+$ etc. ; this one, which we can easily verify ; because if we multiply this infinite series with the denominator $1-2h\cos\theta+h^2$ of the fraction, we find again its numerator, in observing that we have $2\cos n\theta\cos\theta = \cos(n+1)\theta+\cos(n-1)\theta$, whatever is the number n. If h is less than the unit, making abstraction of the sign, this series will be convergent, and the fraction will be rigorously equal to its development prolonged to the infinity ; by cause of

$$1-2h\cos\theta+h^2 = (1-h)^2+4h\sin^2\frac{1}{2}\theta,$$

[10] we will have hence, in this hypothesis,

$$\frac{1-h^2}{(1-h)^2+4h\sin^2\frac{1}{2}\theta} = 1+2\sum h^n\cos n\theta \ ;$$

the sum $\sum$ extending to all the values of the entire number n, from $n=1$ up to $n=\infty$. [11] Whatever are the function $f(\theta)$ and the real constant α, we will have hence also

$$\int_0^\pi \frac{(1-h^2)f(\theta)d\theta}{(1-h)^2+4h\sin^2\frac{1}{2}(\theta-\alpha)} = \int_0^\pi f(\theta)d\theta + 2\sum h^n\int_0^\pi f(\theta)\cos n(\theta-\alpha)d\theta.$$

Let g be a quantity positive and infinitely small ; this equation will subsist more in putting $h=1-g$, because it holds for all value of h less than the unit. For all values

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[8] (⇓) cf. Navier [15].

[9] (⇓) The members of commision in 1807 are Lagrange, Laplace, Monge and Lacroix. cf. [3, 4, 9].

[10] (⇓) The right hand side of the expression (6) is reduced as follows :

$$(1-h)^2+4h\sin^2\frac{1}{2}\theta = 1+h^2-2h+4h\sin^2\frac{1}{2}\theta = 1+h^2-2h\Big(1-2\sin^2\frac{1}{2}\theta\Big) = 1+h^2-2h\cos\theta$$

.

[11] (⇓) This paragraph is expressed by Poisson's phylosophy.

finite of n, we will have $h^n = (1-g)^n = 1$; for the infinite values of this exponent, h^n will be able to differ from the unit, however, in integrating by parts,

$$\int_0^\pi f(\theta)\cos n(\theta-\alpha)d\theta = \frac{1}{n}f(\theta)\sin n(\theta-\alpha) - \frac{1}{n}\int \frac{df(\theta)}{d\theta}\sin n(\theta-\alpha) \ ;$$

so that if $f(\theta)$ never turn to be infinite, between the limits $\theta = 0$ and $\theta = \pi$, nor for these limits, the integral $\int_0^\pi f(\theta)\cos n(\theta-\alpha)d\theta$ which multiply h^m, will evaporate for $n = \infty$; from the above it results that we will be always able to replace h^2 with the unit under the sign $\sum$. At the numerator of the fraction composed under the sign f, we will have$1 - h^2 = 2g$, in neglecting g^2 with respect to $2g$; in the second term of the denominator, we will be able to put the unit instead of h or $1-g$; and, with this manner, we will have

$$(1.2)_{3.3.3} \qquad \frac{1}{2}\int_0^\pi f(\theta)d\theta + \sum\int_0^\pi f(\theta)\cos n(\theta-\alpha)d\theta = \int_0^\pi \frac{gf(\theta)d\theta}{g^2 + 4\sin^2\frac{1}{2}(\theta-\alpha)}. \qquad (6)$$

The coefficient of $d\theta$ under this last integral is infinitely small. except for the values of θ infinitely few different from α, which make its denominator infinitely small ; this integral is hence infinitely small or null, such that the difference $\theta - \alpha$ is a finite quantity ; this one will hold in all the extent of the integration, when we will suppose $\alpha < 0$, or $\alpha > \pi$; hence all the times when the constant α will fall beyond the limits zero and π, we will have the equation

$$(2)_{3.3.3} \qquad \frac{1}{2}\int_0^\pi f(\theta)d\theta + \sum\int_0^\pi f(\theta)\cos n(\theta-\alpha)d\theta = 0. \qquad (7)$$

If, on the contrary, we have $\alpha > 0$ and $< \pi$, there will be the values of θ which differ infinitely few from α ; in putting hence $\theta = \alpha + u, \quad d\theta = du$, the integral of which it acts will evaporate more for the finite value of u, however, no more for the values infinitely small of this variable, positive or negative ; in the regard of this one, we will have

$$f(\theta) = f(\alpha), \qquad \sin\frac{1}{2}(\theta-\alpha) = \frac{1}{2}u \ ; \qquad (8)$$

in consequence, the right hand-side of the equation (6) turns into

$$f(\alpha)\int \frac{g\ du}{g^2+u^2},$$

[12] when α fall between zero and π. Consequently, this integral being null for all value of u which never be infinitely small, we can now extend, without altering the value, to the certain values of u, positive or negative, and take it, if we wish, from $u = -\infty$ up to

[12]($\Downarrow$) From the last expression in the (8), we get

$$4\sin^2\frac{1}{2}(\theta-\alpha) = u^2$$

.

$u = \infty$: we will have hence $\int_{-\infty}^{\infty} \frac{g\ du}{g^2+u^2} = \pi$, [13] and finally

$$(3)_{3.3.3} \qquad \frac{1}{2}\int_0^\pi f(\theta)d\theta + \sum \int_0^\pi f(\theta)\cos n(\theta-\alpha)d\theta = \pi f(\alpha). \tag{9}$$

This reasoning will be convenient also at the case where α coincides with one of two limits zero or π ; however, we have $\alpha = 0$ we will be able to give to u only the positive values, and only of the negative values, if we have $\alpha = \pi$, in order that in these two cases, the variable θ (whom) we have made equal to $\alpha + u$, doesn't separate from the limits of the integration. From this manner, the integral relative to u will find reduce to the half of its value, or to $\frac{1}{2}\pi$; and if we represent with β and γ the values of $f(\alpha)$ which respond to $\alpha = 0$ and $\alpha = \pi$, it will result from the above

$$(4)_{3.3.3} \qquad \begin{cases} \frac{1}{2}\int_0^\pi f(\theta)d\theta + \sum \int_0^\pi f(\theta)\cos n\theta d\theta = \frac{1}{2}\pi\beta, \\ \frac{1}{2}\int_0^\pi f(\theta)d\theta + \sum(-1)^n \int_0^\pi f(\theta)\cos n\theta d\theta = \frac{1}{2}\pi\gamma. \end{cases} \tag{10}$$

Now, put $\theta \equiv \frac{\pi x'}{a}$, $\qquad d\theta = \frac{\pi dx'}{a}$; and let be also $\left(\frac{x'}{a}\right) \equiv \varphi(x')$. The quantity x being positive and less than the constant a, putting instead of α, $-\frac{\pi x}{a}$ in the equation (7) (=$(2)_{3.3.3}$) and $\frac{\pi x}{a}$ in the equation (9) (=$(3)_{3.3.3}$); in observing that the limits relative to x' will be zero and a, we will have

$$(5)_{3.3.3} \qquad \begin{cases} \frac{1}{2}\int_0^a \varphi(x')dx' + \frac{1}{a}\sum \int_0^a \varphi(x')dx' \cos\frac{n\pi(x'+x)}{a}dx' = 0, \\ \frac{1}{2}\int_0^a \varphi(x')dx' + \frac{1}{a}\sum \int_0^a \varphi(x')dx' \cos\frac{n\pi(x'-x)}{a}dx' = \varphi(x) ; \end{cases} \tag{11}$$

and in subtracting these two equations in both sides, it turns into

$$\frac{2}{a}\sum\Big(\int_0^a \varphi(x')\ \sin\frac{n\pi x'}{a}dx'\Big)\sin\frac{n\pi x}{a} = \varphi(x) ; \tag{12}$$

[14]

§ **326**. This formula represents the values of the function $\varphi(x)$, for all the values of the variables x, which are positive and less than a, and similarly, for $x = 0$ and $x = a$, when $\varphi(x)$ will be null for these extreme values. It is important to observe that the series indicated with $\sum$, will always finish with being convergent ; because for the very great value of n, the integral relative to x' will turn into a very small quantity, which will diminish more and more along with that n will augment, and which will be completely null for $n = \infty$, as we have seen above by the method of the integration by parts. This remark is necessary and sufficient to justify the adaption that we will make with the precedent formula.

The different formulae with which we can therefore represent in series of periodical quantities, always convergent, from portions of arbitrary functions continuous or discontinuous, are deduced from the equations (11) (=$(5)_{3.3.3}$), which we are going to establish.

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[13]($\Downarrow$)

$$\int_0^\infty \frac{a\ dx}{a^2+x^2} = \begin{cases} \frac{\pi}{2} & a > 0, \\ 0 & 0, \\ -\frac{\pi}{2} & a < 0. \end{cases}$$

cf. Peirce [16, p.67].

[14]($\Downarrow$) The expression (12) is reduced from the formula :

$$\cos\alpha - \cos\beta = -2\sin\frac{1}{2}\Big(\alpha+\beta\Big)\cdot\sin\frac{1}{2}\Big(\alpha-\beta\Big).$$

I will content to give here two of these formulae, which we will be useful in the following ; for greater development on this material, I will refer to my *memoir on the integral calculus* [15] which makes party of the *Journal de l'École Polytechnique*, and where we will find a complete theory of this genre of transformation.

After having jointed the equations (11) (=(5)$_{3.3.3}$) and subtracted the primary from the second, I put there $2l$ instead of a, next, $x+l$ and $x'+l$ instead of x and x', and successively $\varphi(x)$ and $\varphi(x')$ instead of $\varphi(x+l)$ and $\varphi(x'+' l)$; the limits of the integrals relative to x' turn $\pm l$, and these equations are replaces with as follows :

$$\begin{aligned}
\varphi(x) &= \frac{1}{2l}\int_{-l}^{l}\varphi(x')dx' + \frac{1}{l}\sum\Big(\int_{-l}^{l}\varphi(x')\cos\frac{n\pi(x'+l)}{2l}dx'\Big)\cos\frac{n\pi(x+l)}{2l},\\
\varphi(x) &= \frac{1}{l}\sum\Big(\int_{-l}^{l}\varphi(x')\sin\frac{n\pi(x'+l)}{2l}dx'\Big)\sin\frac{n\pi(x+l)}{2l}.
\end{aligned}$$

Let separate each sum $\sum$ into two others, of which the one relates to the even number n, and the other at the odd number n. For this, were i an entire number certain, and put successively $n = 2i$ $n = 2i - 1$; we will have

$$\cos\frac{2i\pi(x+l)}{2l} = (-1)^i\cos\frac{i\pi x}{l}, \qquad \sin\frac{2i\pi(x+l)}{2l} = (-1)^i\sin\frac{i\pi x}{l},$$
$$\cos\frac{(2i-1)\pi(x+l)}{2l} = (-1)^i\sin\frac{(2i-1)\pi x}{2l}, \qquad \sin\frac{(2i-1)\pi(x+l)}{2l} = -(-1)^i\cos\frac{(2i-1)\pi x}{2l},$$

and similarly, for the sines and cosines composed under the signs $\int$; in consequence, we will have

$$(6)_{3.3.3}\qquad \begin{cases}
\varphi(x) = \frac{1}{2l}\int_{-l}^{l}\varphi(x')dx' + \frac{1}{l}\sum\Big(\int_{-l}^{l}\varphi(x')\cos\frac{i\pi x'}{l}dx'\Big)\cos\frac{i\pi x}{l},\\
+\frac{1}{l}\sum\Big(\int_{-l}^{l}\varphi(x')\sin\frac{(2i-1)\pi x'}{2l}dx'\Big)\sin\frac{(2i-1)\pi x}{2l},\\
\\
\varphi(x) = \frac{1}{l}\sum\Big(\int_{-l}^{l}\varphi(x')\sin\frac{i\pi x')}{l}dx'\Big)\sin\frac{i\pi x}{l}\\
+\frac{1}{l}\sum\Big(\int_{-l}^{l}\varphi(x')\cos\frac{(2i-1)\pi x'}{2l}dx'\Big)\cos\frac{(2i-1)\pi x)}{2l}.
\end{cases} \qquad (13)$$

the sum $\sum$ extend to all the values of i, from $i = 1$ up to $i = \infty$. These equations will hold for all the values of x which will be composed between the limits $\pm l$.

Posed thus, if the function $\varphi(x)$ is such that we would have $\varphi(-x) = -\varphi(x)$, it will result from the above

$$\int_{-l}^{l}\varphi(x')dx' = 0, \qquad \int_{-l}^{l}\varphi(x')\cos\frac{i\pi x'}{l}dx' = 0, \qquad \int_{-l}^{l}\varphi(x')\cos\frac{(2i-1)\pi x'}{2l}dx' = 0,$$

and, in addition,

$$\begin{aligned}
2\int_{-l}^{l}2\varphi(x')\sin\frac{i\pi x')}{l}dx' &= 2\int_{0}^{l}2\varphi(x')\sin\frac{i\pi x'}{l}dx',\\
\int_{-l}^{l}\varphi(x')\sin\frac{(2i-1)\pi x'}{2l}dx' &= 2\int_{0}^{l}\varphi(x')\sin\frac{(2i-1)\pi x'}{2l}dx',
\end{aligned}$$

[15]($\Downarrow$) Fr. *Memoires sur le Calcul integral.*

by the suitable method, the second equation of (13) (=$(6)_{3.3.3}$) will coincides with the formula (14) (=$(a)_{3.3.3}$), [16] in changing a into l ; and the primary is reduced to

$$(7)_{3.3.3} \qquad \varphi(x) = \frac{2}{l}\sum\Big(\int_0^l \varphi(x')\sin\frac{(2i-1)\pi x'}{2l}dx'\Big)\sin\frac{(2i-1)\pi x}{2l}. \qquad (15)$$

If, on the contrary, the function $\varphi(x)$ is such that we would have $\varphi(-x) = \varphi(x)$, we will have

$$\int_{-l}^{l}\varphi(x')\sin\frac{(2i-1)\pi x'}{2l}dx' = 0, \qquad \int_{-l}^{l}\varphi(x')\sin\frac{i\pi x')}{l}dx' = 0 \ ;$$

and the other integrals can be extended only from $x = 0$ up to $x = l$, in doubling the resultants. The second equation of (13) (=$(6)_{3.3.3}$) will make in the equation (15) (=$(7)_{3.3.3}$), in putting $l - x$ instead of x, and $\varphi(x)$ instead of $\varphi(l-x)$. The first equation of (13) (=$(6)_{3.3.3}$) will turn into

$$(8)_{3.3.3} \qquad \varphi(x) = \frac{1}{l}\sum\int_0^l \varphi(x')dx' + \frac{2}{l}\sum\Big(\int_0^l \varphi(x')\cos\frac{i\pi x'}{l}dx'\Big)\cos\frac{i\pi x}{l}. \qquad (16)$$

These formulae (15) (=$(7)_{3.3.3}$) and (16) (=$(8)_{3.3.3}$) represent the values of $\varphi(x)$, from $x = 0$ up to $x = l$; those which is deduced, in differentiating them with respect to x, explain, in the same interval, the values of $\frac{d\varphi(x)}{dx}$. The formula (15) (=$(7)_{3.3.3}$) supposes $\varphi(x) = 0$ for $x = 0$, and $\frac{d\varphi(x)}{dx} = 0$ when $x = l$; the formula (16) (=$(8)_{3.3.3}$) requires that we would have $\frac{d\varphi(x)}{dx} = 0$ for $x = 0$ and for $x = l$. When these conditions aren't satisfied, these formulae or their differentials doesn't hold for the extreme values of x.

§ **328**.

- If we put $2a$ instead of a, and successively $x' + a$ and $x + a$ instead of x and x', in the second equation of (11) (=$(5)_{3.3.3}$),
- and if we put $\varphi(a + x) = F(x)$,

we will have

$$F(x) = \frac{1}{4a}\int_{-a}^{a} F(x')dx' + \frac{1}{2a}\sum\int_{-a}^{a} F(x')dx'\cos\frac{n\pi(x'-x)}{2a}dx',$$

for all the values of x composed between $\pm a$. In putting $\frac{\pi}{a} = \varepsilon$, $\frac{n\pi}{2a} = n\varepsilon = u$, where, this equation will be able to be described hence :

$$F(x) = \frac{1}{2\pi}\int_{-a}^{a} F(x')dx' + \frac{1}{\pi}\sum\Big[\int_{-a}^{a} F(x')dx'\cos u(x'-x)dx'\Big]\varepsilon \ ;$$

where, u being a multiple with ε, and the sum $\sum$ extending to all the value of u, from $u = \varepsilon$ up to $u = \infty$. Consequently, if the constant a turn to be infinite, the difference ε of the consecutive values with u will turn to be infinitely small, and the sum $\sum$ will change into a integral taken from $u = \varepsilon$ or $u = 0$, up to $u = \infty$. In putting hence $a = \infty$ and

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[16]($\Downarrow$)

$$(a)_{3.3.3} \qquad dT + \gamma\omega\Big(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}\Big) = 0 \ ; \qquad (14)$$

$\varepsilon = du$, putting the sign $\int$ instead of $\sum$, in suppressing the primary term of the precedent formula, we will have

$$F(x) = \frac{1}{\pi} \int_0^\infty \Big[\int_{-\infty}^\infty F(x') \cos u(x' - x) dx' \Big] du.$$

Fourier has given the first this important formula, which extends to all the values real, positive or negative, of the variable x, and invites, as the preceding, from which it is deduced, to a certain function $F(x)$, continue or discontinue.

3.3. **The conjecture of defect of Proof on Exact differential.**

Poisson [22] comes to a close[17] in appending his opinion about the proof of exact differential in the last pages of [21, pp.173-4]. His conjucture is based on the preceding analysis in [20, pp.382-3].

The proof of the conservation in time and space of an exact differential was discussed by Lagrange, Cauchy, Stokes, and others. The herein-called "Poisson conjecture" in 1831, cited in the Introduction as one of our main motivations for this study, It had its beginnings with the incomplete proof by Lagrange [10]. However, thereafter, Cauchy [2] had presented a proof as early as 1815, while Power [27] and Stokes [29] had tried by other methods.

To date Cauchy's proof is still considered to be the best. Poisson concludes the proof is defect, and even the equation made of tenscendentals satisfy with exact differential at the original time of movement, the equations satisfy no more with it during all the time.

Poisson says : consequently, this one doesn't hold at the regard of the expressions of u, v, w, in series of exponentials and of sines or cosines, which the exponents and the arcs are proportional with t ; and the demonstration being in defect, the proposition is also able to be, and it is effectively in defect, in certain cases which I have found the examples. In each problem, the expressions of u, v, w, which it behaves, satisfy to with the equations relative to the mass and to the surface of the fluid in motion ; in determining suitably the coefficients of exponentials and of the sines or cosines, they represent the initial state and given with all the points of the fluid ; and if the series which results are additionally convergent, this one suffices in order that they enclose the solution of the question, although one of their particular characters weren't always satisfied with the equations which is deduced from that of the motion, with of new differentiations. [25, a.654]

3.4. **The celestial mechanics in conformity to the mathematical physics.**

His top title of his careers in his books is member of institute, of *Bureau of Longitude*, etc. His astronomical theories is backed on this post. We see the today's scientific bases such as knowledges in his books [24, 25] are constructed in the nineteen's century. He says his role for himself is not a physician but a mathematical physician. The mathematical physics is the new learning coming in his days, and he acknowledges as his few fellows in mathematical physician and a rival : Fourier. Poisson's standing point in astronomy is not on the orbitary study but on the mathematical physics of the earth science, which is one of two astronomical schools.

[17] This note's accepted date is signed as Lu : 2/mars/1829.

We cite only § **44**, § **47**-§ **49**, and § **50** (concludion of heat equation).
§ **44**. (Introduction of the heat in motion in all the corps.)

There is always the heat in motion in all the corps, even when of all their points are invariable,

- if each point should have a particular temperature,
- or, even if they should have all a same temperature.

However, the expression *motion of the heat* is taken here, in the another sense ; it signifies the variation of temperature which holds from an instant to the other in a corps which is heated or is cooled ; and the velocity of this motion, in each point of the corps, is the primary differential coefficient of the temperature in respect to the time.

I will call A the corps solid or liquid, homogeneous or heterogeneous, in which we are going to consider the motion of the heat. Let

- M a certain point of A,
- and m a party of this corps, of insensible grade (no. 7),
- and take the point M.

At the end of a certain time t,

- designate with $x,\ y,\ z$, the three rectangular coordinates of M,
- with v the volume of m,
- and with ρ its density,

so that we have $m = v\rho$.. Let also, at the same instant, u the temperature and $\mho$ [18] the velocity of motion of the heat which responds to the point M.

The quantity u will be a function of $t,\ x,\ y,\ z$, dependent on an equation in the partial differences with respect to these four variables, which it will be the pressing problem to form. If A is a corps solid, and which we make neglect its small dilatation, positive or negative, products with the variations of u relative to time, the coordinates $x,\ y,\ z$, according to independent of t, and we will have simply, $\mho = \frac{du}{dt}$.

- If in contrast, we have regard to small displacement of the point M caused from these dilaration,
- or also, if A is a fluid in which the inequality of temperature, or all other cause, hold to the motions of its molecules,

then the coordinates $x,\ y,\ z$, will be the function of t ; and then we will have with the known rules of the differentiation of functions made of functions, [19]

$$(1)_{PS4} \qquad \mho = \frac{du}{dt} + \frac{dx}{dt}\frac{du}{dx} + \frac{dy}{dt}\frac{du}{dy} + \frac{dz}{dt}\frac{du}{dz}\ ; \tag{17}$$

where, expression in which $\frac{dx}{dt},\ \frac{dy}{dt},\ \frac{dz}{dt}$, will be the components of the velocities at the point M, parallel to the axes $x,\ y,\ z$.

§ **47**. Of the point M as center and a radius equal to the linear unit, we describe a spherical surface ; were ds the differential element of this surface, to which gets, the radius of which the direction is that of MM', we will have $dv' = r^2\ dr\ ds$; and according

[18]($\Downarrow$) We use $\mho$, because, in origin, Poisson uses the vertical type of $\propto$ like the opened shape in upper of the numerical letter 8, however, this exact type isn't in our LaTex font system.

[19]($\Downarrow$) sic. The function is repeated.

to the value of the sum $\sum$, the equation (18) [20] will turn out

$$(4)_{PS4} \qquad c\,\frac{du}{dt} = \iint R\ (u' - u)\ dr\ ds\ ; \tag{19}$$

We put here, for abridgment, $\frac{du}{dt}$, instead of $\mho$; however, we will remember that this differential coefficient needs to be taken with relation to t and to all this that depend ; so that it needs to replace $\frac{du}{dt}$ with the formula (17), when the coordinates $x,\ y,\ z$, of the point M will vary with the time.

The limit relative to r of the integral contains in this equation (19) won't be the same, according to the distance of the point M to the surface of A will surpass l or will be shorter than this small segment. In this chapter we will suppose that this were the primary case which holds ; the integral relative to r will come to be hence taken from $r = 0$ to $r = l$, in all directions around M ; we will be able hence to describe the equation (19) under the form

$$(5)_{PS4} \qquad c\,\frac{du}{dt} = \int_0^l \Big[\int R\ (u' - u)\ ds\Big]\ dr\ ; \tag{20}$$

where, the integral in respect to ds will come to be extended to all the elements ds from the spherical surface, and with the reduction in series, we will obtain easily the approximate value.

§ **48**. For these things, I designate with $\alpha,\ \beta,\ \gamma$, the angles which the segment MM' makes with the parallels to the axes $x,\ y,\ z$, traced through the point M. Because of $MM' = r$, then it will result

$$x' - x = r\ \cos\ \alpha, \qquad y' - y = r\ \cos\ \beta, \qquad z' - z = r\ \cos\ \gamma\ ; \tag{21}$$

and, according to the theory of Taylor, we will have

$$\begin{aligned} u' - u \ =\ & \frac{du}{dx}\ r\ \cos\ \alpha + \frac{du}{dy}\ r\ \cos\ \beta + \frac{du}{dz}\ r\ \cos\ \gamma \\ +\ & \frac{1}{2}\frac{d^2u}{dx^2}\ r^2\ \cos^2\ \alpha + \frac{1}{2}\frac{d^2u}{dy^2}\ r^2\ \cos^2\ \beta + \frac{1}{2}\frac{d^2u}{dz^2}\ r^2\ \cos^2\ \gamma \\ +\ & \frac{d^2u}{dx\ dy}\ r^2\ \cos\ \alpha\ \cos\ \beta + \frac{d^2u}{dx\ dz}\ r^2\ \cos\ \alpha\ \cos\ \gamma + \frac{d^2u}{dy\ dz}\ r^2\ \cos\ \beta\ \cos\ \gamma \\ & \text{etc.} \end{aligned} \tag{22}$$

If we develop similarly R in accordance with the power and the products of $u' - u,\ x' - x,\ y' - y,\ z' - z$, we will have also

$$R = V + \left(\frac{dR}{du'}\right)\ (u' - u) + \left(\frac{dR}{dx'}\right)\ (x' - x) + \left(\frac{dR}{dy'}\right)\ (y' - y) + \left(\frac{dR}{dz'}\right)\ (z' - z) + \text{etc.}\ ;$$

where, the parentheses indicating here that it needs to put $u' = u$, $x' = x$, $y' = y$, $z' = z$ according to the differentiation which supposes r invariable, and V designating this which comes at the same time from the function Φ of the (no. 45), so that we have

$$V = \Phi\ (r,\ u,\ u,\ x,\ y,\ z,\ x,\ y,\ z). \tag{23}$$

[20]($\Downarrow$)

$$(3)_{PS4} \qquad c\ \mho = \sum\ \frac{R}{r^2}\ (u' - u)\ v', \tag{18}$$

By means of these developments of R and of $u' - u$, this one of product $\int R\ (u'-u)$ will be composed of terms of this form $H\ r^n\ \cos^i\ \alpha\ \cos^{i'}\ \beta\ \cos^{i''}\ \gamma$; where, H designating a coefficient independent of $\alpha,\ \beta,\ \gamma$, and the exponential $i,\ i',\ i''$, being the number entire and positive which won't be zeros all the three to the times, and of which the exponent n is the sum $i + i' + i''$. Hence in having regard to the limits of the integral relative to ds, we will have $\int \cos^i\ \alpha\ \cos^{i'}\ \beta\ \cos^{i''}\ \gamma\ ds = 0$, here all times which the one of the three numbers $i,\ i',\ i''$, will be odd ; for then this integral will be composed of the elements which will be equal two by two and the contrary sign. When any of number $i,\ i',\ i''$, won't be odd, the integral won't be zero ; the ordinary rules give the exact values, whatever these three number were ; and with this manner, we will have

$$(6)_{PS4} \qquad R\ (u'-u) = H_2\ r^2 + H_4\ r^4 + H_6\ r^6 + \text{etc.}\ ; \tag{24}$$

where, $H_2,\ H_4,\ H_6$, etc., being the differential function of known form, in any of which the partial differences of u will be taken in respect to $x,\ y,\ z$, and are raised to the order marked with its inferior index.

§ **49**. (General equation of the motion of heat) In this hypothesis, we will stop the development of R at the terms dependent on the square of r exclusively. By reason of the system of R in respect to u and u', x and x', y and y', z and z', and of this one, which V represents, we have evidently

$$\left(\frac{dR}{du'}\right) = \frac{1}{2}\frac{dV}{du}, \quad \left(\frac{dR}{dx'}\right) = \frac{1}{2}\frac{dV}{dx}, \quad \left(\frac{dR}{dy'}\right) = \frac{1}{2}\frac{dV}{dy}, \quad \left(\frac{dR}{dz'}\right) = \frac{1}{2}\frac{dV}{dz}\ ;$$

then, it will result hence

$$R = V + \frac{1}{2}\frac{dV}{du}\ (u'-u) + \frac{1}{2}\frac{dV}{dx}\ (x'-x) + \frac{1}{2}\frac{dV}{dy}\ (y'-y) + \frac{1}{2}\frac{dV}{dz}\ (z'-z)\ ; \tag{25}$$

and with this value jointed to that of $u' - u$, [21] we will conclude

$$\begin{aligned} H_2 &= \frac{1}{2}\Big[V\frac{d^2u}{dx^2} + \Big(\frac{dV}{du}\frac{du}{dx} + \frac{dV}{dx}\Big)\frac{du}{dx}\Big]\int \cos^2\alpha\ ds + \frac{1}{2}\Big[V\frac{d^2u}{dy^2} + \Big(\frac{dV}{du}\frac{du}{dy} + \frac{dV}{dy}\Big)\frac{du}{dy}\Big]\int \cos^2\beta\ ds \\ &+ \frac{1}{2}\Big[V\frac{d^2u}{dz^2} + \Big(\frac{dV}{du}\frac{du}{dz} + \frac{dV}{dz}\Big)\frac{du}{dz}\Big]\int \cos^2\gamma\ ds, \end{aligned}$$

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[21]($\Downarrow$) cf. the expression (22), which includes the expressions (21). The expression (25) turns into :

$$R = V + \frac{1}{2}\frac{dV}{du}\ (u'-u) + \frac{1}{2}\frac{dV}{dx}\ r\ \cos\ \alpha + \frac{1}{2}\frac{dV}{dy}\ r\ \cos\ \beta + \frac{1}{2}\frac{dV}{dz}\ r\ \cos\ \gamma,$$

then we get :

$$\begin{aligned} R(u'-u) &= H_2\ r^2 \\ &= \frac{r^2}{2}V\Big(\frac{d^2u}{dx^2}\ \cos^2\ \alpha + \frac{d^2u}{dy^2}\ \cos^2\ \beta + \frac{d^2u}{dz^2}\ \cos^2\ \gamma\Big) \\ &+ \frac{r}{2}\Big[\frac{dV}{du}\ \Big(\frac{du}{dx}\ \cos\ \alpha + \frac{du}{dy}\ \cos\ \beta + \frac{du}{dz}\ \cos\ \gamma\Big) + \Big(\frac{dV}{dx}\ \cos\ \alpha + \frac{dV}{dy}\ \cos\ \beta + \frac{dV}{dz}\ \cos\ \gamma\Big)\Big] \\ &\times\ r\Big(\frac{du}{dx}\ \cos\ \alpha + \frac{du}{dy}\ \cos\ \beta + \frac{du}{dz}\ \cos\ \gamma\Big) \end{aligned}$$

Finally, we get :

$$\begin{aligned} H_2 &= \frac{1}{2}\Big[V\frac{d^2u}{dx^2} + \Big(\frac{dV}{du}\ \frac{du}{dx} + \frac{dV}{dx}\Big)\frac{du}{dx}\Big]\int \cos^2\ \alpha\ ds + \frac{1}{2}\Big[V\frac{d^2u}{dy^2} + \Big(\frac{dV}{du}\ \frac{du}{dy} + \frac{dV}{dy}\Big)\frac{du}{dy}\Big]\int \cos^2\ \beta\ ds \\ &+ \frac{1}{2}\Big[V\frac{d^2u}{dz^2} + \Big(\frac{dV}{du}\ \frac{du}{dz} + \frac{dV}{dz}\Big)\frac{du}{dz}\Big]\int \cos^2\ \gamma\ ds \end{aligned}$$

or more simply

$$\begin{aligned} H_2 &= \frac{1}{2}\Big[V\frac{d^2u}{dx^2}+\frac{dV}{dx}\frac{du}{dx}\Big]\int\cos^2\alpha\ ds+\frac{1}{2}\Big[V\frac{d^2u}{dy^2}+\frac{dV}{dy}\frac{du}{dy}\Big]\int\cos^2\beta\ ds \\ &+ \frac{1}{2}\Big[V\frac{d^2u}{dz^2}+\frac{dV}{dz}\frac{du}{dz}\Big]\int\cos^2\gamma\ ds\ ; \end{aligned}$$

the partial differences [22] of V with respect to $x,\ y,\ z$, being taken in considering u as a function of these three coordinates, and without varying r.

We have additionally

$$\int\cos^2\alpha\ ds=\int\cos^2\beta\ ds=\int\cos^2\gamma\ ds.$$

Moreover, if we call ψ the angle which makes the plane of the segment MM' and of a parallel to the axis of x traced through the point M, with a fixed plane traced through this parallel, we will have

$$ds=\sin\ \alpha\ d\alpha\ d\psi\ ;$$

and the integral relative to ds will come to be extended to all the spherical surfaces, to which this element belongs, then it will result

$$\int\cos^2\alpha\ ds=\int_0^{\pi}\cos^2\alpha\ \sin\ \alpha\ d\alpha\int_0^{2\pi}d\psi=\frac{4\pi}{3}.$$

[23] Hence, in reducing the value of $\int R\ (u'-u)$ at the primary term $H_2\ r^2$ of the series (24), the equation (20) will come to be

$$\begin{aligned} c\frac{du}{dt} &= \frac{2\pi}{3}\Big(\frac{d^2u}{dx^2}\int_0^l V\ r^2\ dr+\frac{du}{dx}\int_0^l\frac{dV}{dx}\ r^2\ dr\Big) \\ &+ \frac{2\pi}{3}\Big(\frac{d^2u}{dy^2}\int_0^l V\ r^2\ dr+\frac{du}{dy}\int_0^l\frac{dV}{dy}\ r^2\ dr\Big) \\ &+ \frac{2\pi}{3}\Big(\frac{d^2u}{dz^2}\int_0^l V\ r^2\ dr+\frac{du}{dz}\int_0^l\frac{dV}{dz}\ r^2\ dr\Big). \end{aligned} \tag{26}$$

The function V being zero for all the values of r longer than l, we will be able to now extend the integral relative to r beyond this limit, and if we want to be until $r=\infty$. If we put also

$$\frac{2\pi}{3}\int_0^{\infty}Vr^2\ dr\equiv k, \tag{27}$$

where, k will be a function of $u,\ x,\ y,\ z$, and we will have

$$\frac{2\pi}{3}\int_0^{\infty}\frac{dV}{dx}\ r^2\ dr=\frac{dk}{dx},\quad \frac{2\pi}{3}\int_0^{\infty}\frac{dV}{dy}\ r^2\ dr=\frac{dk}{dy},\quad \frac{2\pi}{3}\int_0^{\infty}\frac{dV}{dz}\ r^2\ dr=\frac{dk}{dz}\ ;$$

in consequence, the general equation of the motion of the heat will come to be finally

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[22]($\Downarrow$) id.

[23]($\Downarrow$) According to [16, p.41, no.277],

$$\int\cos^m x\ \sin x dx=-\frac{\cos^{m+1}x}{m+1}.$$

$$(7)_{PS4} \qquad c\frac{du}{dt} = \frac{d.k\frac{du}{dx}}{dx} + \frac{d.k\frac{du}{dy}}{dy} + \frac{d.k\frac{du}{dz}}{dz}. \tag{28}$$

When all the points of A get to a stationary state, we will have $\frac{du}{dt} = 0$, and then it will result

$$\frac{d.k\frac{du}{dx}}{dx} + \frac{d.k\frac{du}{dy}}{dy} + \frac{d.k\frac{du}{dz}}{dz} = 0,$$

for the equation relative to this stationary state. [24]

§**50**. (The conclusion of the heat equation.)

The equation (28) coincides with one which I found in years ago in case of a heterogeneous corps [25], however, in supposing hence only that the quantity k depended on the temperature u.

If A is a homogeneous corps,

- k will depend only on u,
- and the equation (28) will be changed as follows :

$$(8)_{PS4} \qquad c\frac{du}{dt} = k\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) + \frac{dk}{du}\left(\frac{du^2}{dx^2} + \frac{du^2}{dy^2} + \frac{du^2}{dz^2}\right). \tag{30}$$

[26] In supposing that this quantity k were independent of u, we could have the equation

$$(9)_{PS4} \qquad c\frac{du}{dt} = k\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right), \tag{31}$$

[27] which we give it ordinarily, and which is reduced, in case of the stationary state, to an equation independent of two quantities c and k, viz.,

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} = 0. \tag{32}$$

[28] After obtained the equation (31), in considering c and k as the constant quantities, we could suppose

- that it will conserve the same form when these quantities will variable,
- that it will suffice to put here for $\frac{k}{c}$ a function given with u,
- and that the equation relative to the stationary state doesn't receive any change.

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[24]($\Downarrow$) The expression (26) is reduced into

$$c\frac{du}{dt} = \left(\frac{d^2u}{dx^2}k + \frac{du}{dx}\frac{dk}{dx}\right) + \left(\frac{d^2u}{dy^2}k + \frac{du}{dy}\frac{dk}{dy}\right) + \left(\frac{d^2u}{dz^2}k + \frac{du}{dz}\frac{dk}{dz}\right) \tag{29}$$

[25]sic. *Journal de l'École Polytechnique*, 19^e cahier, page 87. ($\Downarrow$) Poisson [**?**], [25, p. 677].

[26]($\Downarrow$) Because of $k = k(u)$, from each second terms in the right hand-side of the expression (29) is reduced into

$$\left(\frac{du}{dx}\frac{dk}{dx}\right) + \left(\frac{du}{dy}\frac{dk}{dy}\right) + \left(\frac{du}{dz}\frac{dk}{dz}\right) = \left(\frac{du}{dx}\frac{du}{dx}\frac{dk}{du}\right) + \left(\frac{du}{dy}\frac{du}{dy}\frac{dk}{du}\right) + \left(\frac{du}{dz}\frac{du}{dz}\frac{dk}{du}\right) = \frac{dk}{du}\left(\frac{du^2}{dx^2} + \frac{du^2}{dy^2} + \frac{du^2}{dz^2}\right)$$

[27]($\Downarrow$) The equation (31) means $c\frac{du}{dt} = k\Delta u$, where Δ meaning the Laplacian.

[28]($\Downarrow$) This function u satisfying the equation (32) is called harmonic function. Poisson doesn't mention the harmonic function, however, Poincaré [17, p.237] calls it so.

However, it is seen that these suppositions are never admissible ; the equation (31) and here one which is deduced in case of $\frac{du}{dt} = 0$, were never, in the same case of a homogeneous corps, the exact equation of the motion of the heat and that of the stationary state ; and the formula (30) shows that the independence of partial differences [29] of u of the second order in respect to $x,\ y,\ z$, the true equations need also to contain the square of its partial differences of the primary order. To have regard to displacement of points of A, products with the dilatation and condensations due to variation of the temperature, or from another cause, we will replace, as we mentioned above, $\frac{du}{dt}$ with the formula (??), and the equation (28) will come to be

$$(10)_{PS4} \qquad c\Big(\frac{du}{dt}+\frac{du}{dx}\frac{dx}{dt}+\frac{du}{dy}\frac{dy}{dt}+\frac{du}{dz}\frac{dz}{dt}\Big)=\frac{d.k\frac{du}{dx}}{dx}+\frac{d.k\frac{du}{dy}}{dy}+\frac{d.k\frac{du}{dz}}{dz}. \qquad (33)$$

Here is this equation (33) which we will come to joint, for instance, to the ordinary equations of the motion of liquids, to accomplish it, hence that I proposed already in my *Study of Mechanics* [30] and in a preceding memoir. [31]

In sum, Poisson's [26] is extending his railroadline of analysis of heat motion on the hypothesis based on the molecular radiation. This is the extending effort since the analysis on fluid motion [21], and hydrostatics [23]. cf. [26, p.13]. This comes from the rivalry to Navier and Fourier. Poisson ignores Navier as Arago [1] says, however, to Fourier, Poisson refutes him in the several papers since the paper [18] on the abstract of Fourier's initial work in 1808. Poisson gets to coincide the equation of interior motion of heat with Fourier's as follows, though Poisson's ($(8)_{PS4}$) deducing method is different with Fourier's ($(1)_{PS11}$). (cf. [26, p.347])

5. Conclusions.

We think Poisson's theories in many arenas are to be discussed. In nineteenth century, under the confused situations in the then French Academy, he asserts in persisting for truth, this evokes to his sympathy. We conclude as follows :

1. Poisson' slough is not only from the publishing order of books [32] but also to build the physical mathematics, from the order of old science and order of the Academy of France, including Duhamel's pointing-out [5] on the continuum, such that Poisson's works had the very countless merits to have to *sweep out the all special hypotheses.*
2. It is worth to listening to Poisson's allusion of the demonstrations of the exact differential, for the example of logarithm.

[29]($\Downarrow$) The symbol of partial difference is not used. id.

[30]($\Downarrow$) *Traité de Mécanique*, op. cit. cf. Poisson [19], [24] and [25].

[31]($\Downarrow$) Poisson puts also the another heat equations such as in Chapter 6. entitled : Digression on the integral of the partial differential equations. §76. [26, p.146], or Chapter 11. entitled : Distribution of the heat in certain corps, and specially in a homogeneous sphere primitively heated with a certain manner. §162. [26, p.347] :

$$(1)_{PS11} \qquad \frac{du}{dt}=a^2\left(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\right), \qquad \frac{k}{c}\equiv a^2, \quad \Rightarrow \quad \frac{du}{dt}=a^2\Delta u. \qquad (34)$$

where, u is the heat, k and c are the conductibility and the specific heat of the material. Δ is the Laplacian.

[32]($\Downarrow$) Poisson says [26] in the second book published in 1835 in his academic paradigm of *Mathematical physics*, which is published after the mechanics : in 1833, [24, 25] cf. Table 1.

3. After Fourier's paper 1807, Poisson changes his thinking to Lagrange's expression and to valuate finally the Fourier's originality.
4. Another expressions of capillary action different from Laplace and Gauss are deduced in considering density.
5. Poisson shows radically different method of deduction of heat equation from Fourie, based on a modern style on continuum, and induced to the same formulation with Fourier.

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